Math Class II

Chapter III Differential Equations

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<u>Differential Equations</u>

A differential equation is an equation which involves derivatives. If there is a single independent variable then the derivatives are ordinary derivatives and the equation is called ***Ordinary Differential Equation*** such as:

$\frac{dy}{dx} = x + 5$ ………………………………………………………...Eq 1

$(y^{//})^2 + (y^{/})^3 + 3y = x^2$………………..…………………………Eq 2

If there are two or more independent variables then the derivatives are partial derivatives and the equation is called ***Partial Differential Equation*** such as:

$\frac{\partial z}{\partial x} = z + x\frac{\partial z}{\partial y}$…………………………………………………………,,.Eq 3

$\frac{\partial 2z}{\partial x^2} + \frac{\partial 2z}{\partial y^2} = x^2 + y$ ………………………………………………..Eq 4

The order of the differential equation is the order of the highest derivatives which occurs for example:

$xy^{/} + 3 = 0$ is of the ***First Order***……….....Eq 5

$\frac{d2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$ is of the ***Second Order***…………Eq 6

$y^{///} + 2(y^{//})^2 + y^{/} = cosx$ is of the ***Third Order*** …...……Eq 7

The degree of the differential equation which can occur written as a polynomial in the derivative is the degree of the highest order derivatives which occurs. Equations 1,3,4,5,6 are of the first degree and equation 2 is of the second degree.

<u>Solution of Differential Equation</u>

A function y=f(x) is said to be a solution of a differential equation if the differential equation is satisfied when y and its derivatives are replaced throughout by f(x) and its derivatives. For example if A, B and C are three arbitrary constants then:

$y = Ax^2 + Bx + c$ is a solution of the differential equation $\frac{d3y}{dx^3} = 0$.

Thus a solution of a differential equation will be a number of arbitrary constants (such as A, B & C above) equal to the order of the differential equation ($\frac{d3y}{dx^3}$ above). This solution is called a general solution.

Problems:

Show that each function is the solution of the accompanying differential equation:

Ex1) $xy^{//} - y^{/} = 0$ $\quad\quad$ $y = x^2 + 3$

$y^{/} = 2x$

$y^{//} = 2$

By substituting in differential equation we get:

$x(2) - 2x = 0$ $\quad\quad$ $0 = 0$

Ex2) $xy^{//} - y^{/} = 0$ $\quad\quad$ $y = C_1x^2 + C_2$

$y^{/} = 2C_1x$

$y^{//} = 2C_1$

By substituting in differential equation we get:

$x(2C_1) - 2C_1x = 0$ $\quad\quad$ $0 = 0$

First Order Differential Equation:

1) Variable Seperable

A first order differential equation can be solved by integration if it is possible to collect all **y** terms with **dy** and all **x** terms with **dx** that is if it possible to write the differential equation in the form:

$f(y)dy + g(x)dx = 0$

Then the general solution is

$\int f(y)dy + \int g(x)dx = C$

Where C is any arbitrary constant

Ex1) Solve the following differential equation:

$\frac{x^2}{x-1}dx + \frac{y^2}{y+1}dy = 0$

Solution: Divide x^2 by (x-1) and y^2 by (y+1) we get

$\left(x + 1 + \frac{1}{x-1}\right)dx + \left(y - 1 + \frac{1}{y+1}\right)dy = 0$

$\int \left[x + 1 + \frac{1}{x-1}\right]dx + \int \left[y - 1 + \frac{1}{y+1}\right]dy = c$

$\frac{x^2}{2} + x + \ln(x-1) + \frac{y^2}{2} - y + \ln(y+1) = c$

$x^2 + 2x + y^2 - 2y + 2\ln(x-1) + 2\ln(y+1) = 2c$

$x^2 + 2x + 1 + y^2 - 2y + 1 + 2\ln(x-1)(y+1) = 2c + 2$

$(x+1)^2 + (y-1)^2 + 2\ln(x-1)(y+1) = 2c + 2$

$(x+1)^2 + (y-1)^2 + 2\ln(x-1)(y+1) = c_1$

Ex2) Solve the following differential equation:

$x(2y-3)dx + (x^2+1)dy = 0$

Solution: multiply the equation by $\frac{1}{(x^2+1)(2y-3)}$

$\frac{x}{x^2+1}dx + \frac{1}{2y-3}dy = 0$

$\frac{1}{2}\int \frac{2x}{x^2+1}dx + \frac{1}{2}\int \frac{2}{2y-3}dy = c$

$\frac{1}{2}ln(x^2+1) + \frac{1}{2}ln(2y-3) = c$

$ln\sqrt{x^2+1} + ln\sqrt{2y-3} = c$

$ln\sqrt{x^2+1}\sqrt{2y-3} = c$

$\sqrt{x^2+1}\sqrt{2y-3} = e^c = c_1$

$(x^2+1)(2y-3) = c_2$

Ex3) Solve the following differential equation:

$x^2(y^2+1)dx + y\sqrt{x^3+1}dy = 0$

Solution: multiply the equation by $\frac{1}{(y^2+1)\sqrt{x^3+1}}$

$\frac{x^2}{\sqrt{x^3+1}}dx + \frac{y}{y^2+1}dy = 0$

$\frac{1}{3}\int \frac{3x^2}{\sqrt{x^3+1}}dx + \frac{1}{2}\int \frac{2y}{y^2+1}dy = c$

$\frac{1}{3}(x^3+1)^{1/2} \times 2 + \frac{1}{2}ln(y^2+1) = c$ then multiply by 6

$4\sqrt{x^3+1} + 3\ln(y^2+1) = 6c = c_1$

Ex4) Solve the following differential equation:

$\frac{dy}{dx} = e^{x-y}$

Solution: $\frac{dy}{dx} = \frac{e^x}{e^y}$

$e^x dx = e^y dy$

$\int e^x dx = \int e^y dy$ +c

$e^y + c = e^x$

$e^y = e^x - c$

2) **Homogeneous**

An equation of the form

$\frac{dy}{dx} = F(\frac{y}{x})$……………………………………………………………Eq.1

Is called a homogeneous equation in which the variables **cannot be separated** but can be transformed by a change of variable into an equation where the variable can be separated,

Assuming $v = \frac{y}{x}$ …………………………………………………..Eq.2

Then $y = vx$

Differentiating above equation we get

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ by substituting y and $\frac{dy}{dx}$ in Eq.1 we get

$v + x\frac{dv}{dx} = F(v)$ …………………………………………………..Eq.3

Eq. 3 is 1st order differential equation variable separable then

$x\frac{dv}{dx} = F(v) - v$ then $\frac{dx}{x} = \frac{dv}{F(v)-v}$

$\frac{dx}{x} - \frac{dv}{F(v)-v} = 0$

$\boldsymbol{\frac{dx}{x} + \frac{dv}{v-F(v)} = 0}$ …………………………………………………..Eq.4

After integration and solution of Eq.4 we must make a back substitution of the variables $v = \frac{y}{x}$ to get the final answer.

Solve the following differential equations

Ex1) $x^2dy = (xy - y^2)dx$

Solution:

$x^2dy = (xy - y^2)dx$

$\frac{dy}{dx} = \frac{xy-y^2}{x^2} = \frac{y}{x} - \frac{y^2}{x^2}$

Let $v = y/x$ then $F(v) = v - v^2$

$$\boldsymbol{\frac{dx}{x}+\frac{dv}{v-F(v)}=0}$$

$$\frac{dx}{x}+\frac{dv}{v^2}=0$$

$$\int\frac{dx}{x}+\int\frac{dv}{v^2}=C$$

$$lnx-\frac{1}{v}=C$$

$$lnx-\frac{x}{y}=C$$

Ex2) $\left(xe^{y/x}+y\right)dx-xdy=0$

Solution:

$$xdy=\left(xe^{y/x}+y\right)dx$$

$$\frac{dy}{dx}=e^{y/x}+\frac{y}{x}$$

Let $v=y/x$ then $F(v)=e^v+v$

$$\boldsymbol{\frac{dx}{x}+\frac{dv}{v-F(v)}=0}$$

$$\frac{dx}{x}+\frac{dv}{v-e^v-v}=0$$

$$\frac{dx}{x}-e^{-v}dv=0$$

$$\int\frac{dx}{x}-\int e^{-v}dv=C$$

$$lnx+e^{-v}=C$$

$$lnx+e^{-y/x}=C$$

Ex3) $\left(ysin\frac{x}{y}-xcos\frac{x}{y}\right)dy+ycos\frac{x}{y}dx=0$

Solution:

$$ycos\frac{x}{y}dx=\left(xcos\frac{x}{y}-ysin\frac{x}{y}\right)dy$$

$$\frac{dx}{dy} = \frac{x\cos\frac{x}{y} - y\sin\frac{x}{y}}{y\cos\frac{x}{y}}$$

$$\frac{dx}{dy} = \frac{x}{y} - \frac{\sin\frac{x}{y}}{\cos\frac{x}{y}}$$

Let $v = x/y$ then $F(v) = v - \frac{\sin v}{\cos v}$

$$\boldsymbol{\frac{dy}{y} + \frac{dv}{v - F(v)} = 0}$$

$$\frac{dy}{y} + \frac{dv}{v - (v - \frac{\sin v}{\cos v})} = 0$$

$$\frac{dy}{y} + \frac{dv}{\frac{\sin v}{\cos v}} = 0$$

$$\int \frac{dy}{y} + \int \frac{\cos v dv}{\sin v} = C$$

$lny + \ln(\sin v) = C$ then $\ln(y\sin v) = C$

$$y\sin v = e^{c} = C_1$$

$$y\sin\frac{x}{y} = C_1$$

3) First Order Linear

A linear differential equation is one in which each of the terms are of degree one or zero, where in computing the degree of the term we add the exponent of the dependent variable and any of its derivatives that occur, thus

$\frac{d2y}{dx^2}$ is of the 1st degree

$y\frac{dy}{dx}$ is of the 2nd degree

A first order linear differential equation can be put into the following form

$\frac{dy}{dx} + Py = Q$ P & Q are functions of x ………………………………Eq. 1

In order to solve Eq.1 we must find the function of $\rho = \rho(x)$ where

$\boldsymbol{\rho = e^{\int Pdx}}$ which is called the integrating factor of Eq.1 and the final solution will be

$\boldsymbol{\rho y = \int \rho Q dx + C}$

Notes:

1) In solving problems involving linear differential equations the following properties of logarithms may be useful
   $e^{lnA} = A$, $e^{mlnA} = e^{lnA^m} = A^m$, $e^{n+mlnA} = e^n A^m$
2) Linear differential equations may be separable or homogeneous and can be solved accordingly
3) If linear equations is linear in x and $\frac{dx}{dy}$ then it can solved by interchanging x and y in above equations.

Solve the following differential equations:

Ex1) $2\frac{dy}{dx} - y = e^{x/2}$

Solution:

Here $P = -1$ and $Q = e^{x/2}$

Then $\boldsymbol{\rho = e^{\int Pdx}} = e^{\int -dx} = e^{-x}$

$\boldsymbol{\rho y = \int \rho Q dx + C}$

$e^{-x}y = \int e^{-x}e^{x/2}dx + C$

$e^{-x}y = \int e^{-x/2}dx + C = -2\int -\frac{1}{2}e^{-x/2}\, dx + C$

$e^{-x}y = -2e^{-x/2} + C$ multiply both sides by e^x

$y = Ce^x - 2e^{x/2}$

Ex2) $xdy + ydx = ydy$

Solution:

$x + y\frac{dx}{dy} = y$

$\frac{dx}{dy} + \frac{x}{y} = 1$ it is linear in x

Here $P = \frac{1}{y}$ and $Q = 1$

Then $\boldsymbol{\rho} = \boldsymbol{e}^{\int \boldsymbol{Pdy}} = e^{\int 1/y dy} = e^{lny} = y$

$\boldsymbol{\rho x} = \int \boldsymbol{\rho Q dy} + \boldsymbol{C}$

$yx = \int ydy + C$

$yx = \frac{y^2}{2} + C$

$x = \frac{y}{2} + Cy^{-1}$

Ex3) $(x-1)^3\frac{dy}{dx} + 4(x-1)^2 y = x + 1$

Solution:

$\frac{dy}{dx} + \frac{4y}{x-1} = \frac{(x+1)}{(x-1)^3}$

Here $P = \frac{4}{x-1}$ and $Q = \frac{x+1}{(x-1)^3}$

Then $\boldsymbol{\rho} = \boldsymbol{e}^{\int \boldsymbol{Pdx}} = \boldsymbol{e}^{\int \frac{4}{x-1}\boldsymbol{dx}} = \boldsymbol{e}^{\boldsymbol{4ln}(\boldsymbol{x}-\boldsymbol{1})} = \boldsymbol{e}^{\boldsymbol{ln}(\boldsymbol{x}-\boldsymbol{1})^{\boldsymbol{4}}} = (\boldsymbol{x}-\boldsymbol{1})^{\boldsymbol{4}}$

$\boldsymbol{\rho y} = \int \boldsymbol{\rho Q dx} + \boldsymbol{C}$

$(x-1)^4 y = \int (x-1)^4\frac{x+1}{(x-1)^3}dx + C = \int (x-1)(x+1)dx + C$

$(x-1)^4 y = \int (x^2-1)dx + C = \frac{x^3}{3} - x + C$

4) <u>First Order Ecaxt</u>

An equation that can be written in the form:

$M(x,y)dx + N(x,y)dy = 0$ ………………………………………Eq. 1

And having the properties that

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ …………………………………………………………....Eq.2

Is said to be exact differential such as

$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = Mdx + Ndy = 0$ …………………………....Eq.3

Where $M = \frac{\partial f}{\partial x}$ and $N = \frac{\partial f}{\partial y}$

Or $\frac{\partial M}{\partial y} = \frac{\partial 2f}{\partial y \partial x}$ and $\frac{\partial N}{\partial x} = \frac{\partial 2f}{\partial x \partial y}$

Sometimes an equation may be seen to be exact equation after regrouping of its terms for example:

$x^2 dx + y^2 dy - ydx - xdy = 0$

$(x^2 - y)dx + (y^2 - x)dy = 0$ is an exact equation since

$\frac{\partial M}{\partial y} = -1$ and $\frac{\partial N}{\partial x} = -1$

Now if Eq.1 is exact differential then

$df = 0$ then the solution is $f(x, y) = C$

<u>Note:</u> A first order differential equation of the form

$M(x, y)dx + N(x, y)dy = 0$ ……………………………………………………Eq.4

Can be made exact by multiplication by a suitable integrating factor $\rho(x, y)$ such an integrating factor has the property that

$\frac{\partial}{\partial y}[\rho(x, y)M(x, y)] = \frac{\partial}{\partial x}[\rho(x, y)N(x, y)]$

Ex1) Solve the following differential equation by using the integrating factor $\frac{1}{x^3}$

$(x + 2y)dx - xdy = 0$

Solution:

$M = (x + 2y)$ and $N = -x$ then

$\frac{\partial M}{\partial y} = 2$ and $\frac{\partial N}{\partial x} = -1$

Not exact, now multiply the equation by $\frac{1}{x^3}$

$\left(\frac{x+2y}{x^3}\right) dx - \frac{x}{x^3} dy = 0$

$\left(\frac{1}{x^2} + \frac{2y}{x^3}\right) dx - \frac{1}{x^2} dy = 0$

$M = \left(\frac{1}{x^2} + \frac{2y}{x^3}\right)$ and $N = -\frac{1}{x^2}$

$\frac{\partial M}{\partial y} = \frac{2}{x^3}$ and $\frac{\partial N}{\partial x} = \frac{2}{x^3}$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then exact

$N = \frac{\partial f}{\partial y} = -\frac{1}{x^2}$

$f = -\int \frac{1}{x^2} dy + k(x)$

$f = \frac{-y}{x^2} + k(x)$

$\frac{\partial f}{\partial x} = \frac{2y}{x^3} + k^/(x) = M = \frac{1}{x^2} + \frac{2y}{x^3}$

$k^/(x) = \frac{1}{x^2}$

$k(x) = \int \frac{1}{x^2} dx + C = -\frac{1}{x} + C$

Then $f = \frac{-y}{x^2} - \frac{1}{x} + C = 0$

$\frac{y}{x^2} + \frac{1}{x} = C$

Ex2) Solve the equation: $(2xy + y^2)dx + (x^2 + 2xy - y)dy = 0$

Solution:

$M = (2xy + y^2)$ and $N = (x^2 + 2xy - y)$

Then $\frac{\partial M}{\partial y} = 2x + 2y$ and $\frac{\partial N}{\partial x} = 2x + 2y$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then exact

$M = \frac{\partial f}{\partial x} = 2xy + y^2$

$f = \int (2xy + y^2)dx + k(y)$

$f = x^2y + y^2x + k(y)$

$\frac{\partial f}{\partial y} = x^2 + 2yx + k^/(y) = N$

$x^2 + 2yx + k^/(y) = x^2 + 2xy - y$

$k^/(y) = -y$

$k(y) = -\frac{y^2}{2}$

$f = x^2y + y^2x - \frac{y^2}{2} = C$

Second Order Ordinary Differential Equations

Special Types of Second Order Equations

Certain types of second order differential equations such as:

$F\left(x, y, \frac{dy}{dx}, \frac{d2y}{dx^2}\right) = 0$ ……………………………………………….Eq. 1

Can be reduced to first order equation by a suitable change of variables:

Type 1

Equation with dependent variable missing then Eq. 1 has the form:

$F\left(x, \frac{dy}{dx}, \frac{d2y}{dx^2}\right) = 0$ ………………………………………………….Eq. 2

It can be reduced to a first order equation by a substituting:

$\rho = \frac{dy}{dx}$ then $\frac{d2y}{dx^2} = \frac{d\rho}{dx}$

Then Eq 2 has the form

$F\left(x, \rho, \frac{d\rho}{dx}\right) = 0$

Which is of the 1st order in $\boldsymbol{\rho}$ and it can be solved according to its type

<u>Type 2</u>

Equation with independent variable is missing then Eq 1 has the form

$F\left(y, \frac{dy}{dx}, \frac{d2y}{dx^2}\right) = 0$……………………………………………….Eq. 3

Which it can be solved by substituting

$\rho = \frac{dy}{dx}$ then $\frac{d2y}{dx^2} = \frac{d\rho}{dx} = \frac{d\rho}{dy}\frac{dy}{dx} = \rho\frac{d\rho}{dy}$

Then Eq. 3 takes the form

$F\left(y, \rho, \rho\frac{d\rho}{dx},\right) = 0$……………………………………………….Eq. 3

Which is of the 1st order in $\boldsymbol{\rho}$ and it can be solved according to its type

Ex1): solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

Solution:

Let $\rho = \frac{dy}{dx} \rightarrow \frac{d^2y}{dx^2} = \rho\frac{d\rho}{dy}$

$\rho\frac{d\rho}{dy} + \rho = 0 \rightarrow \frac{d\rho}{dy} + 1 = 0$

$d\rho + dy = 0$

$\int d\rho + \int dy = c$

$\rho + y = c$

$$\frac{dy}{dx} + y = c \quad \rightarrow \quad \frac{dy}{dx} = c_1 - y$$

$$\frac{dy}{c_1 - y} = dx$$

$$-\ln(c_1 - y) = x + c_2$$

$$\ln(c_1 - y) = -x - c_2$$

$$c_1 - y = e^{-x}.e^{-c_2}$$

$$c_1 - y = c_2\, e^{-x}$$

$$y = c_1 - c_2\, e^{-x}$$

Ex2):

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$$

Sol.

Let $\rho = \frac{dy}{dx} \quad \rightarrow \frac{d^2y}{dx^2} = \frac{d\rho}{dx}$

$$\frac{d\rho}{dx} + x\rho = 0 \quad \rightarrow \quad \frac{d\rho}{\rho} + xdx = 0$$

$$\int \frac{d\rho}{\rho} + \int xdx = c$$

$$\ln\rho + \frac{x^2}{2} = c$$

$$\rho = e^{c - \frac{x^2}{2}}$$

$$\frac{dy}{dx} = e^c.e^{-\frac{x^2}{2}}$$

$$dy = c_1.e^{-\frac{x^2}{2}}\,dx$$

$$\int dy = \int c_1.e^{-\frac{x^2}{2}}\,dx + c_2$$

$y = \int c_1 . e^{-\frac{x^2}{2}} \, dx + c_2$

Ex3): $\frac{d^2y}{dx^2} = 1 + \left(\frac{dy}{dx}\right)^2$

Solution:

Let $\rho = \frac{dy}{dx} \quad \rightarrow \frac{d^2y}{dx^2} = \frac{d\rho}{dx}$

$$\frac{d\rho}{dx} = 1 + \rho^2 \quad \rightarrow \quad \frac{d\rho}{1 + \rho^2} = dx$$

$$\int \frac{d\rho}{1 + \rho^2} = \int dx + c$$

$$\tan^{-1}\rho = x + c \quad \rightarrow \quad \rho = \tan(x + c_1)$$

$\frac{dy}{dx} = \tan(x + c_1)$

$$\int dy = \int \tan(x + c_1) \, dx + c_2$$

$$y = \int \frac{\sin(x + c_1)}{\cos(x + c_1)} dx + c_2$$

$$y = -\ln(\cos(x + c_1)) + c_2$$

linear equations with constant coefficients

An equation of the form

$$\frac{d^ny}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \ldots \ldots \ldots \ldots + a_{n-1} \frac{dy}{dx} + a_n y = f(x) \quad \ldots \ldots (1)$$

Which is linear in y and its derivative is called a linear equation of order n. If f(x) is zero, the equation is called homogeneous, otherwise is called non-homogeneous.

It is convenient to introduce the symbol D to represent the operation of differential on thus.

$Df(x) = \frac{df(x)}{dx} = \frac{dy}{dx} \qquad for\ y, f(x)$

$$D^2 f(x) = D[Df(x)] = \frac{d^2 f(x)}{dx^2} = \frac{d^2 y}{dx^2}$$

$$D^3 f(x) = D[D^2 f(x)] = \frac{d^3 f(x)}{dx^3} = \frac{d^3 y}{dx^3}$$

So that $(D^2 + 4D \ + 3)f(x) = D^2 f(x) + 4Df(x) + 3f(x)$

$$= \frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2} + 4\frac{dy}{dx} + 3y$$

Such a polynomial in D is called the linear differential operator, such polynomial with constant coefficient satisfy basic algebraic laws that make it possible to treat them like ordinary polynomial.

ex:$(D^2 + 4D + 3)f(x) = (D + 3)(D + 1)f(x)$

Linear, second order, homogeneous equations with constant coeffecients

If it is desired to solve

$$\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2} + 2a\frac{dy}{dx} + by = 0 \quad \dots\dots(1) \ \ (a, b \text{ constant})$$

$$(D^2 + 2aD + b)y = 0 \quad \dots\dots(2)$$

And the algebraic equation associated with this differential equation is

$$r^2 + 2ar + b = 0 \qquad \dots\dots(3)$$

Which is obtained by substituting r for D. Thus equations is called the characteristic equation of the differential eq.and the solution eq.(3) leeds to the roots r_1, r_2.then the solution of the differential equation is

Case (1) if $r_1 \neq r_2$ and they are real then

$$\mathrm{y} = \mathrm{c}_1.\mathrm{e}^{\mathrm{r}_1\mathrm{x}} + \mathrm{c}_2.\mathrm{e}^{\mathrm{r}_2\mathrm{x}}$$

Case (2) if $r_1 = r_2 = r$ and they are real then

$$\mathrm{y} = \mathrm{c}_1.\mathrm{e}^{\mathrm{rx}} + \mathrm{c}_2.\mathrm{x}.\mathrm{e}^{\mathrm{rx}}$$

Case (3) if r_1, r_2 are not real (imaginary root)

Where $r_1 = \boldsymbol{\alpha} + \boldsymbol{i}\,\boldsymbol{\beta}$, $r_2 = \boldsymbol{\alpha} - \boldsymbol{i}\,\boldsymbol{\beta}$ then

$y = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$

Ex1): $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$

Solution:

$(D^2 + 2D)y = 0$

$r^2 + 2r = 0 \quad \rightarrow \quad r(r+2)=0 \quad \rightarrow \quad r_1 = 0 \;,\; r_2 = -2$

$y = c_1 . e^{r_1 x} + c_2 . e^{r_2 x}$

$y = c_1 . e^{0} + c_2 . e^{-2x}$

$y = c_1 + c_2 . e^{-2x}$

Ex2): $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y = 0$

Solution:

$(D^2 + 6D + 5)y = 0$

$r^2 + 6r + 5 = 0 \quad \rightarrow \quad (r+5)(r+1) = 0 \quad \rightarrow \quad r_1 = -5 \;,\; r_2 = -1$

$y = c_1 . e^{r_1 x} + c_2 . e^{r_2 x}$

$y = c_1 . e^{-5x} + c_2 . e^{-x}$

Ex3): $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

Solution:

$r^2 + r + 1 = 0 \quad \rightarrow \quad r = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(1)}}{2} = \frac{-1 \pm \sqrt{-3}}{2}$

$r_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i \;,\; r_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

$$\alpha = -\frac{1}{2} \quad , \quad \boldsymbol{\beta} = \frac{\sqrt{3}}{2}$$

$$y = e^{\alpha x}[c_1 \ \cos\beta x + c_2 \ \sin\beta x]$$

$$y = e^{-\frac{1}{2}x}\left[c_1 \ \cos\frac{\sqrt{3}}{2}x + c_2 \ \sin\frac{\sqrt{3}}{2}x\right]$$

linear, second order, non homogeneous equations with constant coefficient

an equation of the form

$$\frac{\mathrm{d^2y}}{\mathrm{dx^2}} + 2a\frac{dy}{dx} + by = f(x) \ \ \ldots\ldots\ldots.(1)$$

Can be solved by first obtaining the general solution of the related homogenous equation:

$$\frac{\mathrm{d^2y}}{\mathrm{dx^2}} + 2a\frac{dy}{dx} + by = 0 \quad \ldots\ldots\ldots.(2)$$

And denote this solution by

$$y_h = c_1u_1(x) + c_2u_2(x) \quad \ldots\ldots\ldots.(3)$$

Where $c_1, \mathrm{and} c_2$ are arbitrary constants and $u_1(x), \ u_2(x)$ are functions of one or more of the following forms $e^{rx}, xe^{rx}, e^{\alpha x}\cos\beta x, e^{\alpha x}\sin\beta x$.

Now by inspection, we may be able to discover one particular function $y=y_p(x)$ that satisfies equation (1). In this case we would be able to solve equation (1) completely as

$$y = y_h + y_p(x)$$

Ex):

Solve the equation $\frac{\mathrm{d^2y}}{\mathrm{dx^2}} + 4\frac{dy}{dx} + 3y = 9$

Solution:

$$\frac{\mathrm{d^2y_h}}{\mathrm{dx^2}} + 4\frac{dy_h}{dx} + 3y_h = 0$$

The characteristic equation is $r^2 + 4r + 3 = 0$

$(r+1)(r+3) = 0 \quad \rightarrow \quad r_1 = -1 \ , \qquad r_2 = -3$

$y_h = c_1 . e^{-x} + c_2 . e^{-3x}$

Now to find a particular solution of the original equation, observe that y=constant would do

$y_p = c \quad \frac{dy_p}{dx} = 0 \quad \frac{d^2y_p}{dx^2} = 0$ substituted in the original equation.

$0 + 4 * 0 + 3y_p = 9 \quad \rightarrow \quad y_p = 3$

The complete solution is $y = y_h + y_p = c_1 . e^{-x} + c_2 . e^{-3x} + 3$

Variation of parameters

There is a general method for finding the solution of the non homogenous eq.(1). Once the general solution of the corresponding homogeneous equation is known. The method is known as the method of variation of parameters. It consists of replacing the constants c_1 and c_2 in eq.(3) by functions $v_1 = v_1(x)$ and $v_2 = v_2(x)$ and requiring that the resulting expression satisfies eq.(1). There are two functions to be determined, and requiring that eq.(1) to be satisfied is only one condition. As a second condition we also require that

$v_1' u_1 + v_2' u_2 = 0 \quad \ldots\ldots\ldots (4)$

Then we have $y = v_1 u_1 + v_2 u_2$

$$\frac{dy}{dx} = v_1 u_1' + v_1' u_1 + v_2 u_2' + v_2' u_2 \qquad (v_1' u_1 + v_2' u_2 = 0)$$

$$\frac{dy}{dx} = v_1 u_1' + v_2 u_2'$$

$$\frac{d^2y}{dx^2} = v_1 u_1'' + v_2 u_2'' + v_1' u_1' + v_2' u_2'$$

If we substitute these expressions into the left hand side of eq.(1) we obtain

$$v_1u_1'' + v_2u_2'' + v_1'u_1' + v_2'u_2' + 2a[v_1u_1' + v_2u_2'] + b[v_1u_1 + v_2u_2]$$
$$= f(x)$$

$$v_1\left[\frac{d^2u_1}{dx^2} + 2a\frac{du_1}{dx} + bu_1\right] + v_2\left[\frac{d^2u_2}{dx^2} + 2a\frac{du_2}{dx} + bu_2\right] + v_1'u_1' + v_2'u_2'$$
$$= f(x)$$

The two bracketed terms are zero since u_1 and u_2 are solutions of the homogeneous equation (2). Hence eq.(1) satisfies if

$$v_1'u_1' + v_2'u_2' = f(x) \quad \dots\dots\dots (5)$$

Equations (4) and (5) may be solved as a pair

$$v_1'u_1 + v_2'u_2 = 0 \quad \dots\dots\dots (4)$$

$$v_1'u_1' + v_2'u_2' = f(x) \quad \dots\dots\dots (5)$$

The unknown functions v_1' and v_2' may be determined using Cramer's rule.

Cramer's rule

A system of linear equations may be solved using Cramer's rule. For example the equations.

$$a_{11}x + a_{12}y = b_1$$

$$a_{21}x + a_{22}y = b_2$$

Can be solved when the determinant of the coefficient matrix

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$$

$$x = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{D} \quad , \quad y = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{D}$$

For equations (4) and (5) the solutions are

$$\left.\begin{aligned} v_1' &= \frac{\begin{vmatrix} 0 & u_2 \\ f(x) & u_2' \end{vmatrix}}{\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}} = \frac{-u_2 f(x)}{D} \\ v_2' &= \frac{\begin{vmatrix} u_1 & 0 \\ u_1' & f(x) \end{vmatrix}}{\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}} = \frac{u_1 f(x)}{D} \end{aligned}\right\} \qquad \ldots\ldots\ldots(6)$$

Then v_1 and v_2 can be found by integration.

Summary: In applying the method of variation of parameters to solve the equation

$\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2} + 2a\frac{dy}{dx} + by = f(x) \quad \ldots\ldots\ldots$ (1), we work directly with equations in (6). It is not necessary to derive them. The steps are

1-solve the associated homogeneous equation $\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2} + 2a\frac{dy}{dx} + by = 0$ to find the functions u_1 and u_2.

2-calculate D, v_1', and v_2' from (6). Where $D = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}$

3- integrate v_1', and v_2' to find v_1 and v_2

4-write down the general solution of (1) as $y = v_1 u_1 + v_2 u_2$

Ex1): solve $\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2} + 4\frac{dy}{dx} + 3y = 9$

Solution:

First Solve the associated homogeneous equation $\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2} + 4\frac{dy}{dx} + 3\mathrm{y} = 0$

The characteristic equation is $r^2 + 4r + 3 = 0$

$(r+3)(r+1) = 0 \quad \rightarrow \quad r_1 = -3 \ , \qquad r_2 = -1$

$u_1(x) = e^{-3x} \ , \ u_2(x) = e^{-x} \quad y_h = c_1.u_1(x) + c_2.u_2(x)$

$$D = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = \begin{vmatrix} e^{-3x} & e^{-x} \\ -3e^{-3x} & -e^{-x} \end{vmatrix} = -e^{-4x} - (-3e^{-4x}) = 2e^{-4x}$$

$$v_1' = \frac{-u_2 f(x)}{D} = \frac{-e^{-x} * 9}{2e^{-4x}} = -\frac{9}{2}e^{3x}$$

$$v_2' = \frac{u_1 f(x)}{D} = \frac{e^{-3x} * 9}{2e^{-4x}} = \frac{9}{2}e^{x}$$

$$v_1 = \int v_1' dx = \int -\frac{9}{2}e^{3x}dx = -\frac{3}{2}e^{3x} + c_1$$

$$v_2 = \int v_2' dx = \int \frac{9}{2}e^{x}dx = \frac{9}{2}e^{x} + c_2$$

$$y = v_1 u_1 + v_2 u_2 = \left(-\frac{3}{2}e^{3x} + c_1\right)e^{-3x} + \left(\frac{9}{2}e^{x} + c_2\right)e^{-x}$$

$$y = -\frac{3}{2} + c_1 e^{-3x} + \frac{9}{2} + c_2 e^{-x}$$

$$y = c_1 e^{-3x} + c_2 e^{-x} + 3$$

Ex2): solve $\frac{d^2y}{dx^2} - y = x$

Solution:

The homogeneous equation $\frac{d^2y}{dx^2} - y = 0$

The characteristic equation is $r^2 - 1 = 0 \;\rightarrow\; r^2 = 1 \;\rightarrow\; r_1 = +1$, $r_2 = -1$

$y_h = c_1 . u_1(x) + c_2 . u_2(x)$, $y_h = c_1\, e^{x} + c_2\, e^{-x}$

$$D = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix} - e^{0} - e^{0} = -1 - 1 = -2$$

$$v_1' = \frac{-u_2 f(x)}{D} = \frac{-e^{-x} * x}{-2} = \frac{x}{2}e^{-x}$$

$$v_2' = \frac{u_1 f(x)}{D} = \frac{e^{x} * x}{-2} = -\frac{x}{2}e^{x}$$

$$v_1 = \int v_1' dx = \int \frac{x}{2}e^{-x}dx = \frac{1}{2}\int x e^{-x}dx = \frac{1}{2}[-xe^{-x} - e^{-x} + c_1]$$

(by parts)

$v_2 = \int v_2' dx = \int -\frac{x}{2} \mathrm{e}^{\mathrm{x}} \mathrm{dx} = -\frac{1}{2} \int x \mathrm{e}^{-\mathrm{x}} \mathrm{dx} = -\frac{1}{2} [x\mathrm{e}^{\mathrm{x}} - \mathrm{e}^{\mathrm{x}} + \mathrm{c}_2]$
(by parts)

$$y = v_1 u_1 + v_2 u_2 = \frac{1}{2} [-x\mathrm{e}^{-\mathrm{x}} - \mathrm{e}^{-\mathrm{x}} + \mathrm{c}_1] \mathrm{e}^{\mathrm{x}} - \frac{1}{2} [x\mathrm{e}^{\mathrm{x}} - \mathrm{e}^{\mathrm{x}} + \mathrm{c}_2] \mathrm{e}^{-\mathrm{x}}$$

$$y = -\frac{1}{2} x - \frac{1}{2} + \frac{1}{2} \mathrm{c}_1 \mathrm{e}^{\mathrm{x}} - \frac{1}{2} x + \frac{1}{2} - \frac{1}{2} \mathrm{c}_2 \mathrm{e}^{-\mathrm{x}} \quad \rightarrow \quad y = \frac{1}{2} \mathrm{c}_1 \mathrm{e}^{\mathrm{x}} - \frac{1}{2} \mathrm{c}_2 \mathrm{e}^{-\mathrm{x}} - \mathrm{x}$$

$$y = \mathrm{c}_1 \mathrm{e}^{\mathrm{x}} + \mathrm{c}_2 \mathrm{e}^{-\mathrm{x}} - \mathrm{x}$$

Ex3): solve $\frac{\mathrm{d}^2 \mathrm{y}}{\mathrm{dx}^2} + 2 \frac{dy}{dx} + y = e^{-x}$

Solution:

First Solve the associated homogeneous equation $\frac{\mathrm{d}^2 \mathrm{y}}{\mathrm{dx}^2} + 2 \frac{\mathrm{dy}}{dx} + \mathrm{y} = 0$

The characteristic equation is $r^2 + 2r + 1 = 0$

$$(r+1)(r+1) = 0 \quad \rightarrow \quad r_1 = r_2 = -1$$

$$y_h = [c_1 . x + c_2] . e^{rx} \quad \rightarrow \quad y_h = [c_1 . x + c_2] . e^{-x}$$

$$y_h = c_1 . x . e^{-x} + c_2 . e^{-x}$$

$$u_1(x) = x e^{-x} \ , \ u_2(x) = e^{-x}$$

$$D = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = \begin{vmatrix} \mathrm{xe}^{-\mathrm{x}} & \mathrm{e}^{-\mathrm{x}} \\ -\mathrm{xe}^{-\mathrm{x}} + \mathrm{e}^{-\mathrm{x}} & -\mathrm{e}^{-\mathrm{x}} \end{vmatrix} = \mathrm{xe}^{-\mathrm{x}} . (-\mathrm{e}^{-\mathrm{x}}) - \mathrm{e}^{-\mathrm{x}} (-\mathrm{xe}^{-\mathrm{x}} + \mathrm{e}^{-\mathrm{x}})$$

$$\mathrm{D} = \mathrm{e}^{-2\mathrm{x}} [-x - 1 + x] = -\mathrm{e}^{-2\mathrm{x}}$$

$$v_1' = \frac{-u_2 f(x)}{D} = \frac{-e^{-x} * e^{-x}}{-e^{-2x}} = +1$$

$$v_2' = \frac{u_1 f(x)}{D} = \frac{x e^{-x} * e^{-x}}{-e^{-2x}} = -x$$

$$v_1 = \int v_1' dx = \int +1 \, dx = x + c_1$$

$$v_2 = \int v_2' dx = \int -x \mathrm{dx} = -\frac{x^2}{2} + c_2$$

$y = v_1 u_1 + v_2 u_2 = xe^{-x}(x + c_1) + e^{-x}\left(-\frac{x^2}{2} + c_2\right)$

$\text{y} = e^{-x}\left(c_1 x + \frac{x^2}{2} + c_2\right)$

Undetermined coefficients

The method of variation of parameters is a completely general method for solving non homogeneous equations. However, the calculations involved can be complicated, and in special cases there may be easier methods to use. For example we might guess a particular solution for the equation.

Ex1):

$\frac{d^2y}{dx^2} - 4y = e^x$ of the form $\text{y}_\text{p} = \text{A}e^x$

Substitute $\left(\text{y}_\text{p} = \text{A}e^x\right)$ with its second derivative $\frac{d^2\text{y}_\text{p}}{dx^2} = \text{A}e^x$

Into the differential equation

$\text{A}e^x - 4\text{A}e^x = e^x \quad \rightarrow \quad -3\text{A}e^x = e^x \quad A = -\frac{1}{3}$

$\text{y}_\text{p} = -\frac{1}{3}e^x$

To find y_h we have $\frac{d^2y}{dx^2} - 4y = 0$

$r^2 - 4 = 0$

$r^2 = 4 \;\rightarrow\; r_1 = +2, \qquad r_2 = -2$

$y_h = c_1\, e^{2x} + c_2\, e^{-2x}$

$y = y_h + y_p = c_1\, e^{2x} + c_2\, e^{-2x} - \frac{1}{3}e^x$

The equation $\frac{d^2y}{dx^2} - y = 2x^2 + 5$ has a particular solution of the form

$y_p = cx^2 + Dx + E$, we substitute this polynomial and its second derivative

$\left(\frac{d^2\mathrm{y_p}}{dx^2} = 2C\right)$ into the differential equation

$2C - (Cx^2 + Dx + E) = 2x^2 + 5 \quad \rightarrow \quad -Cx^2 - Dx + 2C - E$
$= 2x^2 + 5$

$C = -2$, $\quad D = 0$, and $2C - E = 5 \quad \rightarrow \quad -2 * 2 - E = 5 \quad \rightarrow \quad E$
$= -9$

$\mathrm{y_p} = -2x^2 + 0 * x - 9 = -2x^2 - 9$

To find y_h we have $\frac{d^2y}{dx^2} - y = 0$

$r^2 - 1 = 0$

$r^2 = 1 \quad \rightarrow \quad r_1 = +1, \qquad r_2 = -1$

$y_h = c_1\, e^x + c_2\, e^{-x}$

$y = y_h + y_p = c_1\, e^x + c_2\, e^{-x} - 2x^2 - 9$

Ex2): solve the equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} = \sin x$ by the method of undetermined coefficients.

Solution:

To find y_h we have $\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$

$r^2 - r = 0 \quad \rightarrow \quad r(r-1) = 0$

$r_1 = +1, \qquad r_2 = 0$

$y_h = c_1\, e^x + c_2$

Now try a particular solution of the form $y_p = A \sin x$

$\frac{\mathrm{d}y_p}{\mathrm{dx}} = \mathrm{A}\cos\mathrm{x} \quad \rightarrow \quad \frac{d^2y_p}{dx^2} = -A\sin x$

$-A\sin x - \mathrm{A}\cos\mathrm{x} = \sin x \quad \rightarrow \quad A = -1$ and A
$= 0 \quad$ eq. (1)has no solution

Try a particular solution of the form $y_p = A\sin x + B\cos\mathrm{x}$

$$\frac{\mathrm{d}y_p}{\mathrm{dx}} = \mathrm{A}\cos\mathrm{x} - B\sin x$$

$$\frac{d^2y_p}{dx^2} = -A\sin x - \mathrm{B}\cos\mathrm{x} \quad \text{sub. in eq. (1)}$$

$$-A\sin x - B\cos\mathrm{x} - (A\cos X - B\sin x) = \sin x$$

$$(B - A)\sin x - (A + B)\cos x = \sin x$$

$$B - A = 1$$

$$A + B = 0$$

$$B = \frac{1}{2} \quad , \quad A = -\frac{1}{2}$$

$$y_p = -\frac{1}{2}\sin x + \frac{1}{2}\cos x$$

$$y = y_h + y_p = c_1\, e^x + c_2\; - \frac{1}{2}\sin x + \frac{1}{2}\cos x$$

Ex3): solve the equation $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = x + 2$ by the method of undetermined coefficients.

Solution:

To find y_h we have $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0$

$$r^2 + 4r + 5 = 0 \quad \rightarrow r = \frac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2} = \frac{-4 \pm \sqrt{-4}}{2}$$

$$= \frac{-4 \pm 2\sqrt{-1}}{2}$$

$$r = -2 \pm i \quad \rightarrow \quad \alpha = -2, and\ \beta = 1$$

$$y_h = e^{\alpha x}[c_1\cos\beta x + c_2\ \sin\beta x]$$

$$y_h = e^{-2x}[c_1\cos x + c_2\ \sin x]$$

Now try a particular solution of the form $y_p = Ax + B$

$$\frac{dy_p}{dx} = A \quad \rightarrow \quad \frac{d^2 y_p}{dx^2} = 0$$

$$0 + 4A + 5Ax + 5B = x + 2$$

$$5A = 1 \quad \rightarrow \quad A = \frac{1}{5}$$

$$4A + 5B = 2 \quad \rightarrow \quad \frac{4}{5} + 5B = 2 \quad \rightarrow \quad B = \frac{6}{25}$$

$$y_p = \frac{x}{5} + \frac{6}{25}$$

$$y = y_h + y_p = e^{-2x}[c_1 \cos x + c_2 \sin x] + \frac{x}{5} + \frac{6}{25}$$

In general the following table may be helpful to assume y_p

TABLE 16.1 The method of undetermined coefficients for selected equations of the form

$$ay'' + by' + cy = G(x).$$

If $G(x)$ has a term that is a constant multiple of . . .	And if	Then include this expression in the trial function for y_p.
e^{rx}	r is not a root of the auxiliary equation	Ae^{rx}
	r is a single root of the auxiliary equation	Axe^{rx}
	r is a double root of the auxiliary equation	Ax^2e^{rx}
$\sin kx$, $\cos kx$	k is not a root of the auxiliary equation	$B\cos kx + C\sin kx$
$px^2 + qx + m$	0 is not a root of the auxiliary equation	$Dx^2 + Ex + F$
	0 is a single root of the auxiliary equation	$Dx^3 + Ex^2 + Fx$
	0 is a double root of the auxiliary equation	$Dx^4 + Ex^3 + Fx^2$

Higher order linear equations with constant coefficients

The method of solving homogeneous and non homogeneous second order linear differential equations with constant coefficients can be extended to equations of higher order. The characteristic algebraic equation associated with the differential equation

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ \dots\dots\dots\dots + a_{n-1} \frac{dy}{dx} + a_n y = f(x)$$

$$(D^n + a_1 D^{n-1} + \ \dots\dots\dots. + a_{n-1} D + a_n) y = f(x) \quad \dots\dots\dots.(1)$$

$$r^n + a_1 r^{n-1} + \ \dots\dots\dots. + a_{n-1} r + a_n = 0 \qquad \dots\dots\dots..(2)$$

If its roots $r_1, r_2, \dots\dots\dots r_n$ are all distinct, the solution of the homogeneous equation related to eq.(1) is

$$y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \ \dots\dots\dots.. + c_n e^{r_n x}$$

Pairs of complex conjugate roots $\alpha \pm i\beta$ can be grouped together, and the corresponding part of y_h can be written in terms of the functions

$e^{\alpha x} \cos \beta x$ and $e^{\alpha x} \sin \beta x$

Variation of parameters

If the general solution of the homogeneous equation is

$$y_h = c_1 u_1 + c_2 u_2 + \ \dots\dots\dots.. + c_n u_n$$

$$v_1' u_1 + v_2' u_2 = 0 \quad \dots\dots\dots(4)$$

$$v_1' u_1' + v_2' u_2' = f(x) \quad \dots\dots\dots(5)$$

Then $y = v_1 u_1 + v_2 u_2 \dots\dots\dots \ v_n u_n$ will be a solution of the non homogeneous equation (1), provided that

$$v_1' u_1 \ + \ v_2' u_2 \ + \ \dots\dots\dots\dots\dots \ + v_n' u_n \ = 0$$

$$v_1' u_1' \ + \ v_2' u_2' \ + \ \dots\dots\dots\dots\dots \ + v_n' u_n' = 0$$

$$v_1' u_1'' \ + \ v_2' u_2'' \ + \ \dots\dots\dots\dots\dots + v_n' u_n'' = 0$$

$$\vdots \qquad \vdots \qquad\qquad \vdots$$

$$v_1' u_1^{(n-2)} + v_2' u_2^{(n-2)} + \ \dots \dots \dots + v_n' u_n^{(n-2)} = 0$$

$$v_1' u_1^{(n-1)} + v_2' u_2^{(n-1)} + \ \dots \dots \dots + v_n' u_n^{(n-1)} = f(x)$$

These equations may be solved for $v_1', v_2', \dots \dots .., v_n'$ by Cramer's rule, and the results is integrated to give $\mathrm{v}_1, \mathrm{v}_2, \dots \dots \dots,$ and v_n

Ex): solve $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$

Solution:

$$D^3y - 3D^2y + 2Dy = 0$$

$$(D^3 - 3D^2 + 2D)y = 0$$

$$r^3 - 3r^2 + 2r = 0$$

$$r(r^2 - 3r + 2) = 0 \quad \rightarrow \quad r(r-1)(r-2) = 0$$

$$r_1 = 0, \qquad r_2 = 1, \ \ r_3 = 2$$

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + c_3 e^{r_3 x}$$

$$y = c_1 e^{0x} + c_2 e^{x} + c_3 e^{2x}$$

$$y = c_1 + c_2 e^{x} + c_3 e^{2x}$$

Ex): solve $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = e^x$

Solution:

$$r^3 - 3r + 2 = 0$$

$$r^3 - r - 2r + 2 = 0$$

$$r(r^2 - 1) - 2(r-1) = 0 \quad \rightarrow \quad r(r-1)(r+1) - 2(r-1) = 0$$

$$(r-1)(r^2 + r - 2) = 0 \quad \rightarrow \quad (r-1)(r-1)(r+2) = 0$$

$$r_1 = 1, \qquad r_2 = 1, \ \ r_3 = -2$$

$$y_h = (c_1 + c_2 x)e^{x} + c_3 e^{-2x}$$

Try $y_p = Ae^x \quad \rightarrow \quad y_p' = y_p'' = y_p''' = Ae^x$

Substitute into eq.(1)

$Ae^x - 3Ae^x + 2Ae^x = e^x \rightarrow 0 = e^x$ [eq.(1) has no solution of the form $y_p = Ae^x$]

Try $y_p = Axe^x \quad \rightarrow$ homework

Try $y_p = Ax^2e^x$

$y_p' = 2Axe^x + Ax^2e^x$

$y_p'' = 2Ae^x + 2Axe^x + 2Axe^x + Ax^2e^x \quad \rightarrow \quad y_p'' = 2Ae^x + 4Axe^x + Ax^2e^x$

$y_p''' = 2Ae^x + 4Ae^x + 4Axe^x + 2Axe^x + Ax^2e^x$

$y_p''' = 6Ae^x + 6Axe^x + Ax^2e^x$

substitute in (1)

$(6Ae^x + 6Axe^x + Ax^2e^x) - 3(2Axe^x + Ax^2e^x) + 2(Ax^2e^x) \quad \rightarrow \quad A = \frac{1}{6}$

$y_p = \frac{1}{6}x^2e^x$

$y = y_h + y_p = (c_1 + c_2x)e^x + c_3e^{-2x} + \frac{1}{6}x^2e^x$

$y = \left(c_1 + c_2x + \frac{1}{6}x^2\right)e^x + c_3e^{-2x}$

Ex): solve $\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 7$

Solution:

$r^4 - 4r^3 + 6r^2 - 4r + 1 = 0$

$[r^4 - 2r^3 + r^2] + [-2r^3 + 4r^2 - 2r] + [r^2 - 2r + 1] = 0$

$r^2[r^2 - 2r + 1] - 2r[r^2 - 2r + 1] + [r^2 - 2r + 1] = 0$

$$[r^2 - 2r + 1][r^2 - 2r + 1] = 0 \quad \rightarrow \quad (r-1)(r-1)(r-1)(r-1) = 0$$

$r_1 = 1, \quad r_2 = 1, \; r_3 = 1, \quad r_4 = 1$

$y_h = (c_1 x^{m-1} + c_2 x^{m-2} + \; \dots \dots . + c_m) e^{rx}$

$y_h = (c_1 x^{4-1} + c_2 x^{4-2} + \; c_3 x^{4-3} + c_4) e^{1*x}$

$y_h = (c_1 x^3 + c_2 x^2 + \; c_3 x + c_4) e^{x}$

Try $y_p = A \quad \rightarrow \quad y_p' = y_p'' = y_p''' = y_p'''' = 0$

Substitute into eq.(1)

$0 - 4 * 0 + 6 * 0 - 4 * 0 + A = 7 \quad \rightarrow \quad A = 7 \quad \rightarrow \quad y_p = 7$

$y = y_h + y_p = (c_1 x^3 + c_2 x^2 + \; c_3 x + c_4) e^{x} + 7$