Math class II

Chapter V
Infinite Series

[Document subtitle]

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Infinite series

Let U(n) be a function of n which has a definite value for all positive integer values of n. Then an expression of the form:

$U(1) + U(2) + U(3) + \ \dots\dots\dots + U(n)$

Is called an infinite series and is written as $\sum_{n=1}^{\infty} U(n) \quad or \quad \sum U(n)$

Examples:

If $U(n) = \frac{1}{n}$ then the series is $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ \dots\dots$

$U(n) = \frac{1}{\sqrt{n}}$ then the series is $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ \dots\dots$

$U(n) = \frac{n}{1+2n}$ then the series is $\frac{1}{1+2} + \frac{1}{1+4} + \frac{1}{1+8} + \dots\dots$

$U(n) = cos\frac{\pi}{n}$ then the series is $cos\,\pi + cos\frac{\pi}{2} + cos\frac{\pi}{3} + \ \dots\dots$

Nature of the series

If we let S_n denotes the sum of the first n term of the series then

$S_n = U(1) + U(2) + U(3) + \ \dots\dots\dots + U(n)$

If $\lim_{n\to\infty} S_n$ is a finite value say $\lim_{n\to\infty} S_n = S$ then the series is said to be convergent

If $\lim_{n\to\infty} S_n$ is infinite value (i.e. $\lim_{n\to\infty} S_n = \pm\infty$ or does not exist), then the series is said to be divergent

Examples): Find the nature of the series

Ex1): $\frac{1}{1*2} + \frac{1}{2*3} + \frac{1}{3*4} + \ \dots\dots + \frac{1}{n(n+1)} + \ \dots\dots$

Sol. let S_n denotes the sum of the first n term

$$S_n = \frac{1}{1*2} + \frac{1}{2*3} + \frac{1}{3*4} + \ \dots\dots + \underbrace{\frac{1}{(n-1)n}}_{(n-1)th\ term} + \underbrace{\frac{1}{n(n+1)}}_{nth\ term}$$

$$S_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ \dots\dots\dots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$S_n = 1 - \frac{1}{n+1}$$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[1-\frac{1}{n+1}\right] = 1$ → The series is convergent

Ex2): $\sum ln\frac{n}{n+1}$

Sol. let S_n denotes the sum of the first n term

$S_n = ln\frac{1}{2} + ln\frac{2}{3} + ln\frac{3}{4} + \dots\dots\dots + ln\frac{n-1}{n} + ln\frac{n}{n+1}$

$= (ln\,1 - ln\,2) + (ln\,2 - ln\,3) + (ln\,3 - ln\,4) + \dots\dots +$

$[ln(n-1) - ln\,n] + [ln\,n - ln(n+1)]$

$S_n = -\,ln(n+1)$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty}[-\,ln(n+1)] = -\infty$ → The series is divergent

The geometric series

The series of the form

$a + ar + ar^2 + ar^3 \dots\dots.. + ar^{n-1}$

Is called a geometric series having the first term (fixed term) is (a) and its common ratio is (r) to find the nature of the series

$S_n = a + ar + ar^2 + ar^3 \dots\dots ar^{n-2} + ar^{n-1} \dots\dots.(1)$ multiply by r

$\mp rS_n = \mp ar \mp ar^2 \mp ar^3 \mp ar^4 \dots\dots \mp ar^{n-1} \mp ar^n \dots\dots.(2)$

Eq. (1) - eq.(2)

$S_n - rS_n = a - ar^n$

$S_n(1-r) = a(1-r^n)$

$S_n = \frac{a(1-r^n)}{(1-r)}$

$if\ |r| < 1\ then\ \lim_{n\to\infty} S_n$ is definite then the series is convergent

$if\ |r| > 1\ then\ \lim_{n\to\infty} S_n$ is infinite then the series is divergent

if $r = -1$ the series is divergent

if $a = 0$ the series is convergent

Ex): find the nature of the series $2+\frac{2}{5}+\frac{2}{5^2}+\frac{2}{5^3}+\ \dots\dots$

Sol.

$a=2\ \ ,\ \ r=\frac{1}{5}\ \ ,|r|<1\quad\rightarrow\quad convergent\ series$

$$S_n=\frac{a(1-r^n)}{(1-r)}=\frac{2\left(1-\left(\frac{1}{5}\right)^n\right)}{\left(1-\frac{1}{5}\right)}=\frac{2}{4/5}=\frac{5}{2}$$

Test by nth term

1) For every convergent series $\sum u(n)$ the $\lim_{n\to\infty} u(n)=0$, but the converse may not be true (not every $\lim_{n\to\infty} u(n)=0$ means the series is convergent)
2) Every $\lim_{n\to\infty} u(n)\neq 0$, then the $\sum u(n)$ is a divergent series.

Note for the following series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ \dots..+\frac{1}{n}$

have $\lim_{n\to\infty} u(n)$=$\lim_{n\to\infty}\frac{1}{n}=0$

$\left[but\ \lim_{n\to\infty} S_n=\infty\right]$ divergent

But the series is divergent one, that means not every $\lim_{n\to\infty} u(n)=0$ is convergent series.

i.e. $\lim_{n\to\infty} u(n)=0$ is inconclusive.

Examples: Determine the nature of the following series

Ex1): $2+\frac{3}{2}+\frac{4}{3}+\frac{5}{4}+\ \dots\dots....$

Sol.

$u(n)=\frac{n+1}{n}$

$$\lim_{n\to\infty} u(n)=\lim_{n\to\infty}\frac{n+1}{n}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)=1$$

$\lim_{n\to\infty} u(n) \neq 0 \quad \rightarrow$ divergent series

Ex2): $\frac{1^2}{1^2+1} + \frac{2^2}{2^2+1} + \frac{3^2}{3^2+1} + \ldots\ldots$

Sol.

$u(n) = \frac{n^2}{n^2+1}$

$\lim_{n\to\infty} u(n) = \lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n^2}}\right) = 1$

$\lim_{n\to\infty} u(n) \neq 0 \quad \rightarrow$ divergent series

H.W. $\frac{1}{1+3^{-1}} + \frac{2}{1+3^{-2}} + \frac{3}{1+3^{-3}} + \ldots\ldots$

Ratio test (for positive term series)

Let $\sum u(n)$ a series of positive terms

$\rho = \lim_{n\to\infty} \frac{u(n+1)}{u(n)}$

1) The series is convergent if $\rho < 1$
2) The series is divergent if $\rho > 1$
3) The series is inconclusive if $\rho = 1$

Notes $n! = 1 * 2 * 3 * \ldots\ldots\ldots..* n$

$(n+1)! = (n+1) * n!$

Test the following series

Ex1): $\sum_{n=1}^{\infty} \frac{n!\, n!}{(2n)!}$

Hare $u(n) = \frac{n!\, n!}{(2n)!}$

$u(n+1) = \frac{(n+1)!\,(n+1)!}{[2(n+1)]!}$

$\rho = \lim_{n\to\infty} \frac{u(n+1)}{u(n)}$

$$\rho = \lim_{n\to\infty} \frac{(n+1)!\,(n+1)!/[2n+2]!}{n!\,n!/(2n)!}$$

$$= \lim_{n\to\infty} \frac{(n+1)*n!\,(n+1)*n!/(2n+2)(2n+1)(2n)!}{n!\,n!/(2n)!}$$

$$= \lim_{n\to\infty} \frac{(n+1)\,(n+1)}{2(n+1)(2n+1)} = \lim_{n\to\infty} \frac{(n+1)}{(4n+2)} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{4+\frac{2}{n}} = \frac{1}{4}$$

$\rho < 1 \quad \to \ \sum u(n)$ is convergent

Ex2): $\sum_{n=1}^{\infty} \frac{2^n+5}{3^n}$

Here $u(n) = \frac{2^n+5}{3^n}$

$u(n+1) = \frac{2^{(n+1)}+5}{3^{(n+1)}}$

$$\rho = \lim_{n\to\infty} \frac{u(n+1)}{u(n)}$$

$$\rho = \lim_{n\to\infty} \frac{\left(2^{(n+1)}+5\right)/3^{(n+1)}}{(2^n+5)/3^n}$$

$$\rho = \lim_{n\to\infty} \frac{\left(2^{(n+1)}+5\right)/3^n*3}{(2^n+5)/3^n} = \lim_{n\to\infty} \frac{(2^n*2+5)}{3(2^n+5)}$$

$\lim_{n\to\infty} \frac{2+\frac{5}{2^n}}{3\left(1+\frac{5}{2^n}\right)} = \frac{2}{3} < 1 \qquad \to$ convergent series

Ex3): $\sum u(n) = \sum_{n=1}^{\infty} n!\,e^{-n}$

Sol.

$$\rho = \lim_{n\to\infty} \frac{u(n+1)}{u(n)} = \lim_{n\to\infty} \frac{(n+1)!\,e^{-(n+1)}}{n!e^{-n}}$$

$$\rho = \lim_{n\to\infty} \frac{(n+1)\,n!\,e^{-n}\,e^{-1}}{n!e^{-n}} = \lim_{n\to\infty} \frac{(n+1)}{e} = \infty$$

$\rho > 1 \quad \to \ \sum u(n)$ is divergent series

Power Series:

McLaurin’s series

$$f(x) = f(0) + xf^{`}_{(0)} + \frac{x^2}{2!}f^{``}_{(0)} + \frac{x^3}{3!}f^{```}_{(0)} + \;\dots\dots + \frac{x^n}{n!}f^{n}_{(0)} + \;\dots.$$

The above series is known as McLaurin’s series for the function $f(x)$

Every continuous function which has value of function and all of its derivatives at x=0 can be represented by McLaurin’s series.

$f(x) = sinx$

$f(x) = cosx$

$f(x) = e^x$

All these functions can be represented by McLaurin’s series

$f(x) = lnx$

$f(x) = 1/x$

All these functions cannot be represented by McLaurin’s series

Ex1): Find the McLaurin’s series for the function $(x) = sin\,mx$.

Sol. $f(x) = f(0) + xf^{`}_{(0)} + \frac{x^2}{2!}f^{``}_{(0)} + \frac{x^3}{3!}f^{```}_{(0)} + \frac{x^4}{4!}f^{````}_{(0)} + \;\dots\dots$ (1)

Here $f(x) = sin\,mx$ $\quad f(0) = 0$

$f^{`}_{(x)} = m\,cos\,mx$ $\quad f^{`}_{(0)} = m * 1 = m$

$f^{``}_{(x)} = -m^2\,sin\,mx$ $\quad f^{``}_{(0)} = 0$

$f^{```}_{(x)} = -m^3\,cos\,mx$ $\quad f^{```}_{(0)} = -m^3 * 1 = -m^3$

$f^{iv}_{(x)} = m^4\,sin\,mx$ $\quad f^{iv}_{(x)} = 0$

$f^{v}_{(x)} = m^5\,cos\,mx$ $\quad f^{v}_{(0)} = m^5 * 1 = m^5$

Hence from (1) we get

$$sin\,mx = 0 + xm + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-m^3) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(m^5) + \ldots \ldots$$

$$sin\,mx = xm - m^3\frac{x^3}{3!} + m^5\frac{x^5}{5!} - m^7\frac{x^7}{7!} + m^9\frac{x^9}{9!} + \ldots \ldots$$

H.w Find McLaurin's for $f(x) = cos\,mx$ then integrate this series.

Taylor's series

$$f(x) = f(a) + (x-a)f^{`}_{(a)} + \frac{(x-a)^2}{2!}f^{``}_{(a)} + \frac{(x-a)^3}{3!}f^{```}_{(a)} + \cdots + \frac{(x-a)^n}{n!}f^{n}_{(a)} + \cdots$$

The above series is known as Taylor's series for the function $f(x)$ about $x=a$.

Every continuous function which has value of function and all of its derivatives at x=a can be represented by Taylor's series at x=a

$f(x) = sinx\ at\ x = \pi/4$

$f(x) = cosx\ at\ x = \pi/4$

$f(x) = e^x\ at\ x = 1$

All these functions can be represented by Taylor's series

$f(x) = secx$ cannot represented by Taylor' s series at $x = \pi/2$

$f(x) = tanx$ cannot represented by Taylor's series at $x = \pi/2$

Ex1): Find the Taylor series for the function $f(x) = ln\,x$ about $x=1$

Sol. $f(x) = f(1) + (x-1)f^{`}_{(1)} + \frac{(x-1)^2}{2!}f^{``}_{(1)} + \frac{(x-1)^3}{3!}f^{```}_{(1)} + \ldots \ldots$

Here $f(x) = ln\,x \qquad f(1) = 0$

$f^{`}_{(x)} = \frac{1}{x} \qquad f^{`}_{(1)} = 1$

$f^{``}_{(x)} = -\frac{1}{x^2} \qquad f^{``}_{(1)} = -1$

$f^{```}_{(x)} = -\frac{2}{x^3} \qquad f^{```}_{(1)} = 2$

$$f^{iv}_{(x)} = -\frac{2*3}{x^4} \qquad f^{iv}_{(1)} = -2*3$$

$$ln\, x = 0 + (x-1) - \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} * 2 + \frac{(x-1)^4}{4!} * (-2*3) + \dots\dots$$

$$ln\, x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots\dots$$

<u>Ex2):</u> Find the Taylor series for the function $f(x) = \sqrt{x}$ about $x = 4$

<u>Sol.</u>

$$f(x) = f(4) + (x-4)f^{`}_{(4)} + \frac{(x-4)^2}{2!}f^{``}_{(4)} + \frac{(x-4)^3}{3!}f^{```}_{(4)} + \dots\dots$$

Here $f(x) = \sqrt{x} \qquad f(4) = 2$

$$f^{`}_{(x)} = \frac{1}{2\sqrt{x}} \qquad f^{`}_{(4)} = \frac{1}{4}$$

$$f^{``}_{(x)} = -\frac{1}{4x^{3/2}} \qquad f^{``}_{(4)} = -\frac{1}{32}$$

$$f^{```}_{(x)} = \frac{3}{8x^{3/2}} \qquad f^{```}_{(4)} = \frac{3}{256}$$

$$\sqrt{x} = 2 + \frac{(x-4)}{4} - \frac{(x-4)^2}{2!}\frac{1}{32} + \frac{(x-4)^3}{3!}\frac{3}{256} + \dots\dots$$

$$\sqrt{x} = 2 + \frac{(x-4)}{4} - \frac{(x-4)^2}{64} + \frac{(x-4)^3}{512} + \dots\dots$$