CHAPTAR 1
Complex Numbers

Definitions.

$$\text{Let } i^2 = -1.$$
$$\therefore\ i = \sqrt{-1}.$$

Complex numbers are often denoted by z.

Just as $\mathbb{R}$ is the set of real numbers, $\mathbb{C}$ is the set of complex numbers. If z is a complex number, z is of the form

$$z = x + iy \in \mathbb{C}, \text{ for some } x, y \in \mathbb{R}.$$

* Rectangular Form *

e.g. $3 + 4i$ is a complex number.

$$z = \underset{\text{real part}}{\underset{\uparrow}{x}} + \underset{\text{imaginary part.}}{\underset{\nwarrow}{iy}}$$

If $z = x + iy$, $x, y \in \mathbb{R}$,

$$\textit{the real part of } z = \Re(z) = \mathrm{Re}(z) = x$$
$$\textit{the imaginary part of } z = \Im(z) = \mathrm{Im}(z) = y.$$

eg. $z = 3 + 4i$

$\Re(z) = 3$

$\Im(z) = 4.$

If $z = x + iy$, then $\overline{z}$ ("*z bar*") is given by

$$\overline{z} = x - iy$$

and is called the *complex conjugate* of z.

eg. If $z = 3 + 4i$, then $\overline{z} = 3 - 4i$.

Example. Solve $x^2 - 2x + 3 = 0$.

$x = \frac{-(-2)\pm\sqrt{(-2)^2-4(1)(3)}}{2(1)} = \frac{2\pm\sqrt{-8}}{2} = \frac{2\pm2\sqrt{-2}}{2} = 1 \pm \sqrt{2}\,i.$ □

The Argand Diagram:

(Note: ordered pairs: eg. 2+j=(2,1) for 2+j=x+jy on (x,y) plane)

Two methods: 1) P(x,y) the point P on the (x,y)-plane

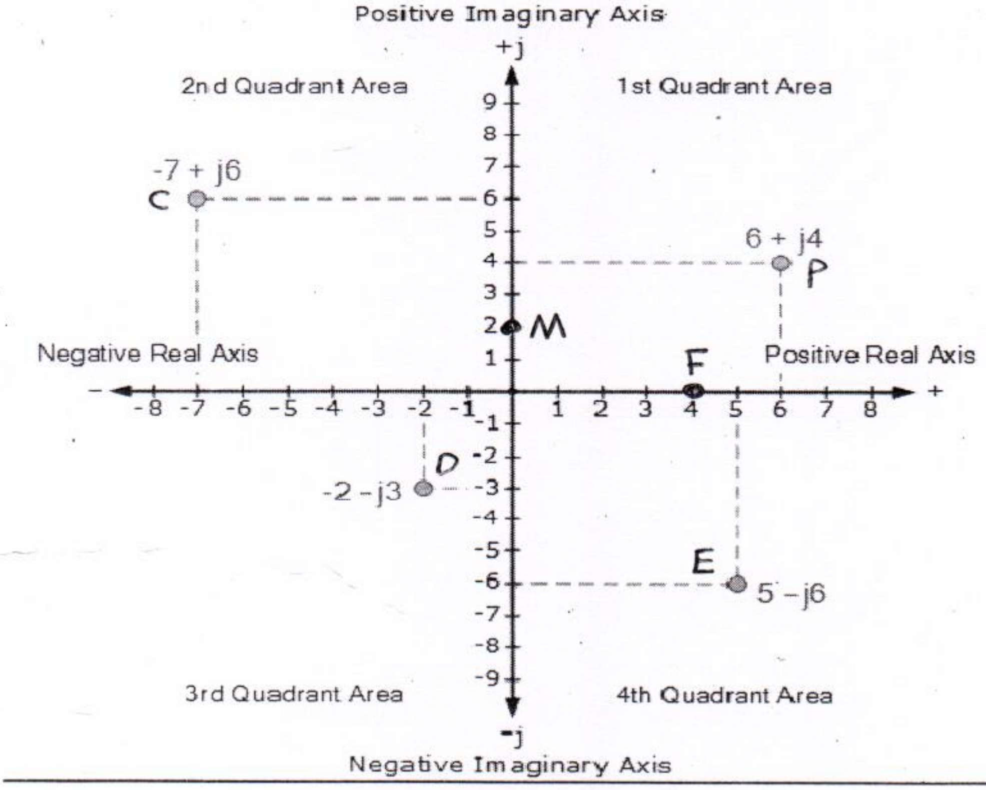
2) Vector $\overrightarrow{OP}$

x-axis is called the real axis.

y-axis is called the imaginary axis.

Example: Plot the following on the Argand diagram:

P=6+j4, C=-7+j6, D=-2-j3, E=5-j6, F=4, M= j2



Complex Arithmetic.

Addition/Subraction.

Example 1. $(2+3i)+(4+i)=6+4i$.
Example 2. $(8-3i)-(-2+4i)=10-7i$.

Multiplication/Division.

Example 1. $(2+3i)(1+2i)=2+4i+3i-6=-4+7i$
Example 2. $(3-2i)(3+2i)=9-(2i)^2=9+4=13$

$\therefore$ when we multiply two complex conjugates, we get a real number.

Example 3. $\frac{2+3i}{1+4i}=\frac{2+3i}{1+4i}\times\frac{1-4i}{1-4i}=\frac{(2+3i)(1-4i)}{(1+4i)(1-4i)}=\frac{2-8i+3i-12i^2}{1-(4i)^2}=\frac{14-5i}{17}$ $=\frac{14}{17}-\frac{5}{17}i$
(*realising* the denominator)
$1+4$

$i=\sqrt{-1}$ $\Rightarrow$ 90° ccw rotation

$i^2=-1$ $\Rightarrow$ 180° ccw rotation

$i^3=i^2\cdot i=-i$ $\Rightarrow$ 270° ccw rotation

$i^4=i^2\cdot i^2=+1$ $\Rightarrow$ 360° ccw rotation

and we have also $\frac{1}{i}=-i$

Theorem. *If two complex numbers are equal then their real parts are equal and their imaginary parts are equal, i.e., if $a+ib=c+id$ where $a,b,c,d \in \mathbb{R}$, then $a=c$ and $b=d$.*

Example 1. Find x, y if $(3+4i)^2 - 2(x-iy) = x+iy$.

Left hand side (LHS) $= 9 - 16 + 24i - 2x + i2y$

[illegible] $7 - 2x + i(24+2y)$

$$\begin{aligned}
\therefore -7 - 2x &= x \\
3x &= -7 \\
x &= -\tfrac{7}{3} \\
\& \ 24 + 2y &= y \\
y &= -24 \quad \square
\end{aligned}$$

Example 2. Find x, y if $\dfrac{x}{1+i} + \dfrac{y}{2-i} = 2+4i$.

$$\begin{aligned}
\text{LHS} &= \frac{x}{1+i} + \frac{y}{2-i} \\
&= \frac{x}{1+i} \times \frac{1-i}{1-i} + \frac{y}{2-i} \times \frac{2+i}{2+i} \\
&= \frac{x(1-i)}{1+1} + \frac{y(2+i)}{4+1} \\
&= \frac{x(1-i)}{2} + \frac{y(2+i)}{5}
\end{aligned}$$

Now $\dfrac{x(1-i)}{2} + \dfrac{y(2+i)}{5} = 2+4i$.

$$\begin{aligned}
\therefore 5x(1-i) + 2y(2+i) &= 20 + 40i \\
5x - i5x + 4y + i2y &= 20 + 40i \\
5x + 4y + i(-5x+2y) &= 20 + 40i
\end{aligned}$$

Equating real and imaginary part,

$$\begin{aligned}
5x + 4y &= 20 \\
-5x + 2y &= 40 \\
\hline
6y &= 60 \\
y &= 10 \\
\& \ \therefore x &= -4. \quad \square
\end{aligned}$$

Solving simultaneously,

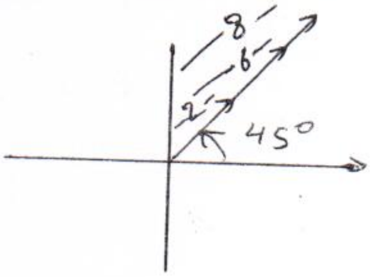
Mathematical operations in the polor form

Rectangular form is best for adding and subtracting complex numbers as we saw above, but polar form is often better for multiplying and dividing. To multiply together two vectors in polar form, we must first multiply together the two modulus or magnitudes and then add together their angles.

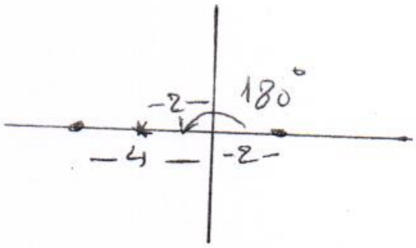
Addition and subtraction:

Addition or subtraction cannot be performed in polar form unless the complex numbers have the same angle θ or differ only by multiples of 180°

1) $2\lfloor 45^\circ + 6\lfloor 45^\circ = 8\lfloor 45^\circ$



2) $2\lfloor 0^\circ + 4\lfloor 180^\circ = 2\lfloor 180^\circ$



Multiplication in Polar Form

$$Z_1 \times Z_2 = A_1 \times A_2 \angle\theta_1 + \theta_2$$

Multiplying together 6 ∠30° and 8 ∠– 45° in polar form gives us.

$$Z_1 \times Z_2 = 6 \times 8 \angle 30^\circ + (-45^\circ) = 48 \angle -15^\circ$$

Division in Polar Form

Likewise, to divide together two vectors in polar form, we must divide the two modulus and then subtract their angles as shown.

$$\frac{Z_1}{Z_2} = \left(\frac{A_1}{A_2}\right) \angle\theta_1 - \theta_2$$

$$\frac{Z_1}{Z_2} = \left(\frac{6}{8}\right) \angle 30^0 - (-45^0) = 0.75 \angle 75^0$$

Converting between Rectangular Form and Polar Form

In the rectangular form we can express a vector in terms of its rectangular coordinates, with the horizontal axis being its real axis and the vertical axis being its imaginary axis or j-component. In polar form these real and imaginary axes are simply represented by "A ∠θ". Then using our example above, the relationship between rectangular form and polar form can be defined as.

Converting Polar Form into Rectangular Form, (P→R)

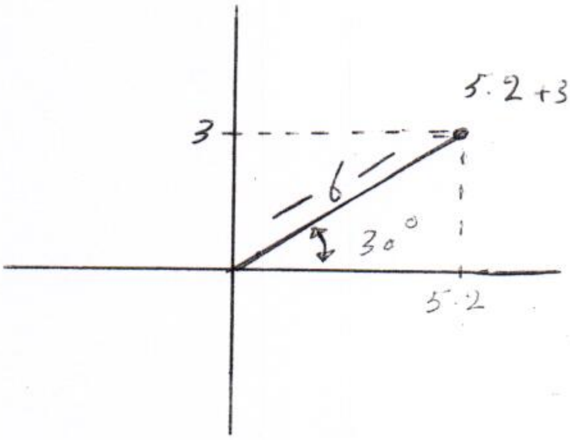
$$6\angle 30^{o} = x + jy$$

However,

$$x = A.\cos\theta \quad y = A.\sin\theta$$

Therefore,

$$\begin{aligned} 6\angle 30^{0} &= (6\cos\theta) + j(6\sin\theta) \\ &= (6\cos 30^{o}) + j(6\sin 30^{o}) \\ &= (6 \times 0.866) + j(6 \times 0.5) \\ &= 5.2 + j3 \end{aligned}$$



We can also convert back from rectangular form to polar form as follows.

Converting Rectangular Form into Polar Form, (R→P)

$$(5.2 + j3) = A\angle\theta$$

$$\text{where:} \quad A = \sqrt{5.2^2 + 3^2} = 6$$

$$\text{and} \quad \theta = \tan^{-1}\frac{3}{5.2} = 30^{o}$$

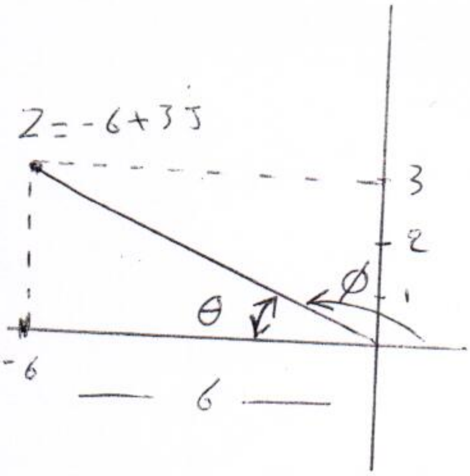
$$\text{Hence,} \quad (5.2 + j3) = 6\angle 30^{0}$$

Example:

Convert the following from rectangular to polar form

$$Z = -6 + 3j$$

Solution:



$$r = \sqrt{(-6)^2 + (3)^2} = \sqrt{45} = 6.71$$

$$\theta = \tan^{-1}\frac{3}{-6} = 26.57^\circ$$

$$>>> \; \emptyset = 180^\circ - 26.57^\circ = 153.43^\circ$$

$$Z = r\lfloor\emptyset = 6.71\lfloor 153.43^\circ$$

Example:

Convert the following from polar to rectangular .

$$Z = 10\lfloor 230^\circ$$

Solution:

X=r cos θ

=10 cos (230°-180°)

=10 cos 50°

=6.428

y=r sin θ

=10 sin (230°-180°)

=10 sin 50°

=7.66

Z=-6.428-7.66 i