Math class II

Chapter II Vectors

[Document subtitle]

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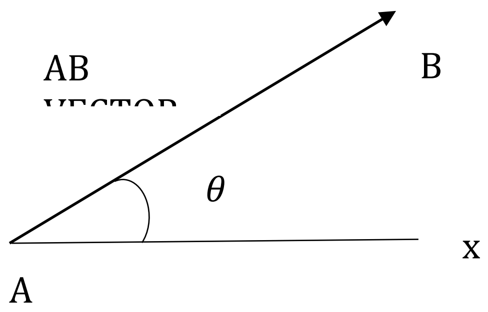
Vectors

Things like length, temperature and time can be described by their quantities only while others like force, speed and displacement must be described by magnitude and direction, where vectors can be used to describe them. Each vector can be represented by a directed line segment. See below figure.



If we use a vector to represent a force, velocity or displacement then the direction of the vector represent the direction of the force, velocity or displacement and the length of the vector represent the force, velocity or displacement magnitude.

In xy plane if point A coordinate is (x1,y1) and B coordinate is (x2,y2) the AB vector can be written as:

$$\overrightarrow{AB} = (x2 - x1)i + (y2 - y1)j$$

Similarly in three dimension if A coordinate is (x1,y1,z1) and B coordinate is (x2,y2,z2) then AB vector can be written as:

$$\overrightarrow{AB} = (x2 - x1)i + (y2 - y1)j + (z2 - z1)k$$

Each vector describes direction and magnitude which equal to the length of vector which can be calculated as:

$$\left|\overrightarrow{AB}\right| = \sqrt{(x2 - x1)^2 + (y2 - y1)^2}$$

And in three dimension:

$$\left|\overrightarrow{AB}\right| = \sqrt{(x2 - x1)^2 + (y2 - y1)^2 + (z2 - z1)^2}$$

If we divide the vector AB by its length we get unit vector as follows:

$$\vec{u} = \frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|} = \cos\theta\ i + \sin\theta\ j$$

Where θ is the angle that the vector AB make with the positive direction of x axis

In three dimensions unit vector can be written as:

$$\vec{u} = \frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|} = \cos\theta\ i + \cos\beta\ j + \cos\gamma\ k$$

Where θ, β and γ are the angles that the vector AB makes with the positive direction of x-axis, y-axis and z-axis respectively.

Basic operations of vectors:

1) Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are said equal if they equal in vector length and having the same direction.
2) Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ can be added or subtracted by adding or subtracting their components as follows:

   Let $\overrightarrow{AB} = ab1\ i + ab2\ j$ and $\overrightarrow{CD} = cd1\ i + cd2\ j$

   The $\vec{V} = \overrightarrow{AB} \pm \overrightarrow{CD} = (ab1 \pm cd1)i + (ab2 \pm cd2)$
3) Multiplication:

a) We can multiply a vector by a magnitude by multiplying each component of the vector by that magnitude.

Let $\overrightarrow{AB} = ab1\ i + ab2\ j$ then if we multiply that vector by quantity F then $F\left(\overrightarrow{AB}\right) = F(ab1)\ i + F(ab2)\ j$

b) Dot (or scalar) product between two vectors :

Let $\overrightarrow{AB} = ab1\ i + ab2\ j$ and $\overrightarrow{CD} = cd1\ i + cd2\ j$

Then $\overrightarrow{AB}.\overrightarrow{CD} = ab1\ .\ cd1 + ab2\ .\ cd2$

Which is quantity and without direction (no vector result)

Note: $i.i = j.j = 1$ while in 3 dimension $i.i = j.j = k.k = 1$

$i.j = j.k = k.i = 0$

Also note $\overrightarrow{AB}.\overrightarrow{CD} = \left|\overrightarrow{AB}\right|\left|\overrightarrow{CD}\right| \cos\theta$

Where θ is the angle between these two vectors

c) Vector product between two vectors:

Let $\overrightarrow{AB} = ab1\ i + ab2\ j + ab3\ k$ and $\overrightarrow{CD} = cd1\ i + cd2\ j + cd3\ k$

Then $\overrightarrow{AB} \times \overrightarrow{CD} = \begin{vmatrix} i & j & k \\ ab1 & ab2 & ab3 \\ cd3 & cd2 & cd3 \end{vmatrix} = \overrightarrow{N}$

Where $\overrightarrow{N}$ is a new vector normal to the plane containing $\overrightarrow{AB}$ and $\overrightarrow{CD}$

Note: $i \times j = k$ and $j \times i = -k$

$j \times k = i$ and $k \times j = -i$

$k \times i = j$ and $i \times k = -j$

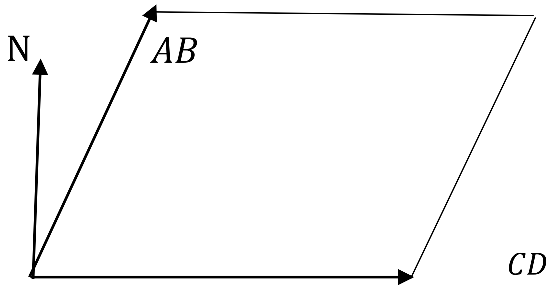
$i \times i = j \times j = k \times k = 0$

$\overrightarrow{AB} \times \overrightarrow{CD} = -\overrightarrow{CD} \times \overrightarrow{AB}$

Also note $\overrightarrow{AB} \times \overrightarrow{CD} = \left|\overrightarrow{AB}\right|\left|\overrightarrow{CD}\right| \sin\theta\ \vec{n}$

Where $\vec{n}$ is a unit normal vector

And the absolute value of $\overrightarrow{N}$ (i.e length of the resultant normal vector) represent the area of parallelogram whose sides are the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$



Ex1) For the vectors $\vec{v} = 2i - 4j + \sqrt{5}k$ and $\vec{u} = -2i + 4j - \sqrt{5}k$.

a) Find $v.u, |v|, |u|$

b) The angle between these two vectors.

c) The scalar component of u in the direction of v

d) The projection of u on v

Sol:

a) $v.u = (2i - 4j + \sqrt{5}k).(-2i + 4j - \sqrt{5}k)$

$$= 2(-2) + (-4)(4) + (\sqrt{5})(-\sqrt{5})$$

$$-4 - 16 - 5 = -25$$

$$|v| = \sqrt{2^2 + (-4)^2 + 5} = \sqrt{4 + 16 + 5} = 5$$

$$|u| = \sqrt{(-2)^2 + 4^2 + 5} = \sqrt{4 + 16 + 5} = 5$$

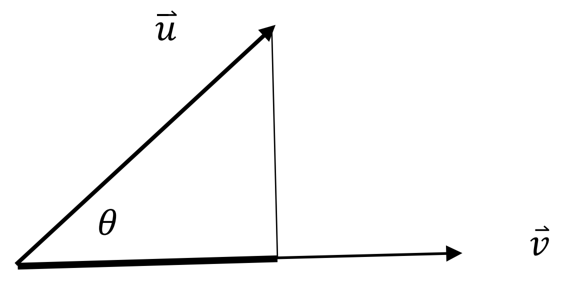
b) $\vec{v}.\vec{u} = |v||u|\cos\theta$

Then $\cos\theta = \frac{\vec{v}.\vec{u}}{|v||u|} = \frac{-25}{5(5)} = -1$

Then $\theta = \cos^{-1} - 1 = \pi = 180^{\cdot}$

c) $|u|\cos\theta = 5(-1) = -5$

Or $|u|\cos\theta = |u|\frac{\vec{v}.\vec{u}}{|v||u|} = \vec{u}\frac{\vec{v}}{|v|} = \frac{-25}{5} = -5$



Length $=|u|cos\theta$

d) projection of u on v= $|u|\cos\theta\frac{\vec{v}}{|v|} = -5\frac{2i-4j+\sqrt{5}k}{5} = -2i + 4j - \sqrt{5}k$

or projection of u on v=$|u|\cos\theta\frac{\vec{v}}{|v|} = |u|\frac{\vec{v}.\vec{u}}{|v||u|}\frac{\vec{v}}{|v|} = \frac{\vec{v}.\vec{u}}{|v|^2}\vec{v}.$

$$= \frac{-25}{5^2}\left(2i - 4j + \sqrt{5}k\right) == -2i + 4j - \sqrt{5}k$$

Ex2) Prove that the following two vectors are orthogonal

$\vec{v} = -i + 3j$ and $\vec{u} = 3i + j$

Solution:

$\vec{v}.\vec{u} = |v||u|\cos\theta$

Then $\cos\theta = \frac{\vec{v}.\vec{u}}{|v||u|} = \frac{-1(3)+3(1)}{|v||u|} = 0$

Then these two vectors are orthogonal

H.W: Find the angles of triangle whose vertices are A=(-1,0), B=(2,1) and C=(1,-2).

Ex3) Find the area of triangle whose vertices are P=(1,-1,0), Q=(2,1,-1) and R=(-1,1,2).

Solution

$\overrightarrow{\mathbf{PQ}} = (\mathbf{2} - \mathbf{1})\mathbf{i} + (\mathbf{1} + \mathbf{1})\mathbf{j} + (-\mathbf{1} - \mathbf{0})\mathbf{k} = \mathbf{i} + \mathbf{2j} - \mathbf{k}$

$\overrightarrow{\mathbf{PR}} = (-\mathbf{1} - \mathbf{1})i + (1 + 1)j + (2 - 0)k = -2i + 2j + 2k$

$$PQ \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix}$$

$= [2(2) + 1(2)]i - [1(2) + 1(-2)]j + [1(2) + 2(2)]k = 6i + 6k$

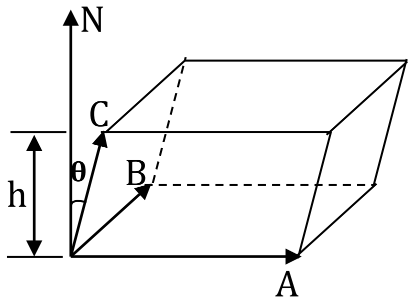
Area of parallelogram whose edges are the vectors PQ and PR is

$|PQ \times PR| = \sqrt{6^2 + (0)^2 + 6^2} = \sqrt{72} = 6\sqrt{2}$

Then the area of triangle is $\frac{6\sqrt{2}}{2} = 3\sqrt{2}$

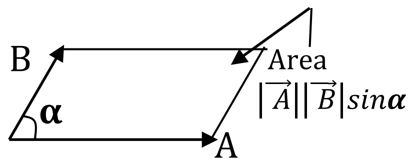
Product of three or more Vectors

Triple scalar product



The product $(\vec{A}\times\vec{B}).\vec{C}$ is called the triple scalar product has the following geometric significance.

The vector $\vec{N}=\vec{A}\times\vec{B}$ is normal to the base of the box determined by the vectors $\vec{A},\vec{B}\ and\ \vec{C}$. The magnitude of $\vec{N}$ equals the area of the base determined by $\vec{A}\ and \vec{B}$. Thus $(\vec{A}\times\vec{B}).\vec{C}=\vec{N}.\vec{C}=|\vec{N}||\vec{C}|\cos\Theta$ is the volume of the box (when neglecting the sign) whose edges $\vec{A},\vec{B}\ and\ \vec{C}$ since $|\vec{N}|=|\vec{A}\times\vec{B}|$ =area of base and $|\vec{C}|\cos\Theta=\pm h=$ $\pm$altitude of the box.



Note: $\vec{A}\times\vec{B}=n|\vec{A}||\vec{B}|sin\boldsymbol{\alpha}$

If $\vec{C}$ and $\vec{A}\times\vec{B}$ lie on the same side of the plane determined by $\vec{A}\ and \vec{B}$, the triple scalar product will be positive. Otherwise the product will be regative. Now if we consider the plane of $\vec{B}\ and\ \vec{C}$ and then the plane of $\vec{C}$ and $\vec{A}$ is the base of the box, then we can see that

$$(\vec{A}\times\vec{B}).\vec{C}=(\vec{B}\times\vec{C}).\vec{A}=(\vec{C}\times\vec{A}).\vec{B} \quad \dots\dots(1)$$

Since the dot nproduct is commutative we also have

$$(\vec{B}\times\vec{C}).\vec{A}=\vec{A}.(\vec{B}\times\vec{C})$$

So that eq.(1) gives the result

$$(\vec{A}\times\vec{B}).\vec{C}=\vec{A}.(\vec{B}\times\vec{C}) \quad \dots\dots(2)$$

Eq.(2) says that the dot and the cross may be interchange in the triple scalar product provided that the multiplication are performed in a way that makes sense.

Thus $(\vec{A}.\vec{B}) \times \vec{C}$ is excluded since $(\vec{A}.\vec{B})$ is a scalar and we never cross a scalar and a vector.

Now if we let $\vec{A} = a_1 i + a_2 j + a_3 k$

$\vec{B} = b_1 i + b_2 j + b_3 k$

$\vec{C} = c_1 i + c_2 j + c_3 k$

Then $\vec{B} \times \vec{C} = \begin{bmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} i - \begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} j + \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} k$

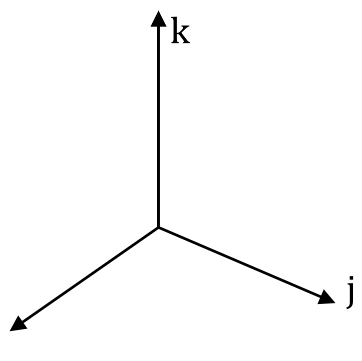
$$\vec{A}.(\vec{B} \times \vec{C}) = a_1 \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} - a_2 \begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} + a_3 \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

Note $i \times j = -j \times i = k$

$j \times k = -k \times j = i$

$k \times i = -i \times k = j$

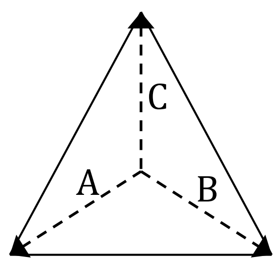
$k \times k = j \times j = i \times i$



Example(1):

Find the volume of tetrahedron with vertices at (0, 0, 0),(1, -1 ,1), (2,1,-2) and (-1, 2, -1).

Sol.



$\vec{A} = i - j + k$

$\vec{B} = 2i + j - 2k$

$\vec{C} = -i + 2j - k$

Volume of the box of edges $\vec{A}, \vec{B}, and\ \vec{C} = (\vec{A} \times \vec{B}).\vec{C} = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -2 \\ -1 & 2 & -1 \end{vmatrix} = 4$

Volume of tetrahedron $= \frac{V}{6} = \frac{4}{6} = \frac{2}{3}$

<u>Example(2):</u> Show that $\vec{A}.(\vec{A} \times \vec{B}) = 0$

<u>Sol.</u>

Let $\vec{A} = a_1 i + a_2 j + a_3 k$

$\vec{B} = b_1 i + b_2 j + b_3 k$

$$\vec{A}.(\vec{A} \times \vec{B}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \begin{matrix} a_1 & a_2 \\ a_1 & a_2 \\ b_1 & b_2 \end{matrix}$$

$= a_1 a_2 b_3 + a_2 a_3 b_1 + a_3 a_1 b_2 - b_1 a_2 a_3 - b_2 a_3 a_1 - b_3 a_1 a_2 = 0$

<u>Triple vector product</u>

The triple vector product $(\vec{A} \times \vec{B}) \times \vec{C}\ and\ \vec{A} \times (\vec{B} \times \vec{C})$ are usually not equal. The product $(\vec{A} \times \vec{B}) \times \vec{C}$ is given by

$$(\vec{A} \times \vec{B}) \times \vec{C} = (\vec{A}.\vec{C})\vec{B} - (\vec{B}.\vec{C})\vec{A} \quad \ldots\ldots(1)$$

Similary

$(\vec{B} \times \vec{C}) \times \vec{A} = (\vec{B}.\vec{A})\vec{C} - (\vec{C}.\vec{A})\vec{B}$

$\vec{A} \times (\vec{B} \times \vec{C}) = -(\vec{B} \times \vec{C}) \times \vec{A}$

$$= -[(\vec{B}.\vec{A})\vec{C} - (\vec{C}.\vec{A})\vec{B}] = (\vec{C}.\vec{A})\vec{B} - (\vec{B}.\vec{A})\vec{C}$$

Or $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A}.\vec{C})\vec{B} - (\vec{A}.\vec{B})\vec{C}$ ………(2)

Example:

For $\vec{A} = 4i - 8j + k$

$\vec{B} = 2i + j - 2k$

$\vec{C} = 3i - 4j + 12k$

1) Find $(\vec{A}.\vec{B}).\vec{C}$ *and* $\vec{A}.(\vec{B}.\vec{C})$
2) Find the volume of the box whose edges $\vec{A}, \vec{B}, and\, \vec{C}$
3) a) find $\vec{A} \times \vec{B}$ and use the result to find $(\vec{A} \times \vec{B}) \times \vec{C}$

   b) find $(\vec{A} \times \vec{B}) \times \vec{C}$ by another method.

Sol.

$(\vec{A}.\vec{B}).\vec{C} = (8 - 8 - 2)(3i - 4j + 12k) = -6i + 8j - 24k$

$\vec{A}.(\vec{B}.\vec{C}) = (4i - 8j + k)(6 - 4 - 24) = -88i + 176j - 22k$

2) $\vec{A}.(\vec{B} \times \vec{C}) = \left| \begin{matrix} 4 & -8 & 1 \\ 2 & 1 & -2 \\ 3 & -4 & 12 \end{matrix} \right| \begin{matrix} 4 & -8 \\ 2 & 1 \\ 3 & -4 \end{matrix} \right|$ =(48+48-8)-(3+32-192)=245

3)a) $\vec{A} \times \vec{B} = \left| \begin{matrix} i & j & k \\ 4 & -8 & 1 \\ 2 & 1 & -2 \end{matrix} \right| \begin{matrix} i & j \\ 4 & -8 \\ 2 & 1 \end{matrix} \right|$

$= (16i + 2j + 4k) - (-16k + i - 8j) = 15i + 10j + 20k$

$(\vec{A} \times \vec{B}) \times \vec{C} = \left| \begin{matrix} i & j & k \\ 15 & 10 & 20 \\ 3 & -4 & 12 \end{matrix} \right| \begin{matrix} i & j \\ 15 & 10 \\ 3 & -4 \end{matrix} \right| = 200i - 120j - 90k$

b) $(\vec{A} \times \vec{B}) \times \vec{C} = (\vec{A}.\vec{C})\vec{B} - (\vec{B}.\vec{C})\vec{A}$

$= (12 + 32 + 12)(2i + j - 2k) - (6 - 4 - 24)(4i - 8j + k)$

$$= 56(2i + j - 2k) + 22(4i - 8j + k) = 200i - 120j - 90k$$

Example:

For the vectors given in the proceeding example, find the triple vector product $\vec{A} \times (\vec{B} \times \vec{C})$

Sol.

$$\vec{B} \times \vec{C} = \begin{vmatrix} i & j & k \\ 2 & 1 & -2 \\ 3 & -4 & 12 \end{vmatrix} \begin{matrix} i & j \\ 2 & 1 \\ 3 & -4 \end{matrix}$$

$$= (12i - 6j - 8k) - (3k + 8i + 24j) = 4i - 30j - 11k$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \begin{vmatrix} i & j & k \\ 4 & -8 & 1 \\ 4 & -30 & -11 \end{vmatrix} \begin{matrix} i & j \\ 4 & -8 \\ 4 & -30 \end{matrix}$$

$$= (88i + 4j - 120k) - (-30i - 44j - 32k) = 118i + 48j - 88k$$

Or $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A}.\vec{C})\vec{B} - (\vec{A}.\vec{B})\vec{C}$

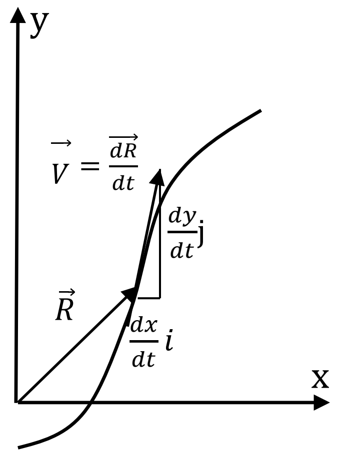
$$= (12 + 32 + 12)(2i + j - 2k) - (8 - 8 - 2)(3i - 4j + 12k)$$

$$= 118i + 48j - 88k$$

Velocity and acceleration

Suppose that point p moves along a curve in the xy-plane and suppose we know its position at any time t. The vector from the origion to P is called the position vector of P. The vector is given by

$\vec{R} = xi + yj$,



$where\ x = f(t)\ and\ y = g(t)$

The velocity vector is given by

$$\vec{V} = \frac{\overrightarrow{dR}}{dt} = \frac{dx}{dt}i + \frac{dy}{dt}j \quad \dots\dots\dots(1)$$

The geometric significance of eq.(1) may be learned by calculating the slope and magnitude of $\frac{\overrightarrow{dR}}{dt}$.

$$Slope\ of\ \frac{\overrightarrow{dR}}{dt} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx}$$

Hence, the velocity vector $\vec{V}$ is tangent to the curve at point P in the direction of motion.

$$The\ magnitude\ of\ \frac{\overrightarrow{dR}}{dt} = \left|\frac{\overrightarrow{dR}}{dt}\right| = \left|\frac{dx}{dt}i + \frac{dy}{dt}j\right|$$

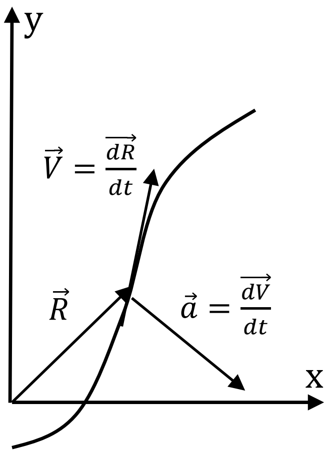
$$|\vec{V}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \left|\frac{dS}{dt}\right|$$

Gives the instantaneous speed at P. Here S represents the arc length along the curve measured from starting point (x_0, y_0). The arc length measured along the curve from P_0 ($x_0(t)$, $y_0(t)$) is given by

$$S(t)=\int\left|\vec{V}\right|dt=\begin{cases}\int_{t_o}^{t}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt & if\ t\geq t_o\\ \int_{t}^{t_o}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt & if\ t<t_o\end{cases}$$

The acceleration vector $\vec{a}$ is obtained from $\vec{V}$ by afurther differentiation.

$$\vec{a}=\frac{\overrightarrow{dV}}{dt}=\frac{d^2x}{dt^2}i+\frac{d^2y}{dt^2}j \qquad \ldots\ldots(2)$$



For a particle of constant mass moving under the action of an applied force $\vec{F}$, Newton's second law states that $\vec{F}=m\vec{a}$

Example(1):

If $\vec{R}=ix+jy=(k\cos wt)i+(k\sin wt)j$ is the vector from the origin to the moving point P(x, y) at time t, find the velocity and acceleration vectors at $t=\frac{\pi}{3w}$ where k and w being positive constantsand find the speed.

Sol.

$$\vec{V}=\frac{\overrightarrow{dR}}{dt}=-(kw\ \sin wt)i+(k\,w\cos wt)j$$

$$\vec{a}=\frac{\overrightarrow{dV}}{dt}=-(kw^2\cos wt)i-(kw^2\sin wt)j$$

$$=-w^2[(k\cos wt)i+(k\sin wt)j=-w^2\vec{R}$$

At $t=\frac{\pi}{3w}$

$$\vec{V} = -\left(kw\ \ sin\frac{\pi}{3}\right)i + \left(k\,w\,cos\frac{\pi}{3}\right)j = -\frac{\sqrt{3}}{2}kwi + \frac{1}{2}kwj = \frac{kw}{2}(-\sqrt{3}i + j)$$

$$The\ speed = \left|\vec{V}\right| = \frac{kw}{2}\sqrt{\left(-\sqrt{3}i\right)^2 + 1^2} = \frac{kw}{2}\sqrt{3+1} = kw$$

$$\vec{a} = -w^2\left[\left(k\,cos\frac{\pi}{3}\right)i + \left(k\,sin\frac{\pi}{3}\right)j\right] = -kw^2[\frac{1}{2}i + \frac{\sqrt{3}}{2}j]$$

$$\vec{a} = -\frac{kw^2}{2}[i + \sqrt{3}j]$$

Example(2):

If the force acts on a particle P of mass (m) is $\vec{F} = -mgj$, where m and g are constants, the particle starts from the origin with velocity

$\overrightarrow{V_0} = (k\,cos\,\alpha)i + (k\,sin\,\alpha)j\ \ at\ t = 0$. Find the position vector $\vec{R} = xi + yj$ from the origin to P at time t..

Sol.

$$\vec{F} = m\vec{a} = m\frac{d^2\vec{R}}{dt^2}$$

$$-mgj = m\frac{d^2\vec{R}}{dt^2}\ \ ,or\ \ -gj = \frac{d^2\vec{R}}{dt^2}$$

$$since\,\vec{V} = \frac{d\vec{R}}{dt}\quad ,\qquad -gj = \frac{d}{dt}\left(\frac{d\vec{R}}{dt}\right) = \frac{d\vec{V}}{dt}$$

$$-gjdt = d\vec{V}\qquad ,or\quad \vec{V} = -gjt + C_1$$

At t=0 $\vec{V} = V_0 = (k\,cos\,\alpha)i + (k\,sin\,\alpha)j$

$$(k\,cos\,\alpha)i + (k\,sin\,\alpha)j = -gj(0) + C_1$$

Hence, $\overrightarrow{C_1} = (k\,cos\,\alpha)i + (k\,sin\,\alpha)j$

And $\vec{V} = -gjt + (k\,cos\,\alpha)i + (k\,sin\,\alpha)j$

$$\vec{V} = \frac{d\vec{R}}{dt} = (k\,cos\,\alpha)i + (k\,sin\,\alpha - gt)j$$

$$d\vec{R} = [(k\,cos\,\alpha)i + (k\,sin\,\alpha - gt)j]\,dt$$

$$\vec{R} = (k\,cos\,\alpha)ti + \left(kt\,sin\,\alpha - \frac{gt^2}{2}\right)j + c_2$$

At $t = 0\,, x = 0\,, y = 0$ (the motion of the particle starts from the origin)

$$\vec{R} = xi + yj = 0$$

$$0 = (k\,cos\,\alpha)(0)i + \left(k(0)\,sin\,\alpha - \frac{g(0)^2}{2}\right)j + c_2 \quad \rightarrow \quad c_2 = 0$$

$$\vec{R} = (k\,cos\,\alpha)ti + \left(kt\,sin\,\alpha - \frac{gt^2}{2}\right)j$$

Note The forgoing equations for velocity and acceleration are suitable for describing the motion of a particle moving on a flat surface (i.e. in the xy-plane). To describe the motion of an object in space, we need three coordinates.

$$\vec{R}(t) = xi + yj + zk$$

$$\vec{V} = \frac{d\vec{R}}{dt} = \frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k$$

$$\vec{a} = \frac{d^2x}{dt^2}i + \frac{d^2y}{dt^2}j + \frac{d^2z}{dt^2}k$$

The length(or magnitude) of the velocity vector is the speed with which the object moves along its path

$$speed = |\vec{V}| = \left|\frac{d\vec{R}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

The arc length of the curve from t=a to t=b is obtained by integrating $|\vec{V}| = \left|\frac{d\vec{R}}{dt}\right|$ from a to b

$$arc\ length = \int_{t=a}^{t=b} |\vec{V}| dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

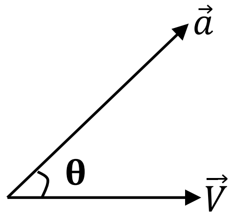
Example(3):

If $\vec{R}(t) = xi + yj + zk = e^t i + e^t \sin t\, j + e^t \cos t\, k$ is the position vector of a particle in space. Find the angle between the velocity and acceleration vectors.

Sol.

$$\vec{V} = \frac{d\vec{R}}{dt} = e^t i + (e^t \sin t + e^t \cos t)j + (e^t \cos t - e^t \sin t)k$$

$$\vec{a} = \frac{d\vec{V}}{dt} = e^t i + (e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t)j + (e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t)k$$



$$\vec{a} = e^t + (2e^t \cos t)j - (2e^t \sin t)k$$

$$\vec{a} \,.\, \vec{V} = |\vec{a}\,||\vec{V}| \cos\theta$$

$$\cos\theta = \frac{\vec{a} \,.\, \vec{V}}{|\vec{a}\,||\vec{V}|}$$

$$|\vec{V}| = \sqrt{e^{2t} + e^{2t}(\sin t + \cos t)^2 + e^{2t}(\cos t - \sin t)^2}$$

$$= e^t\sqrt{1 + \sin^2 t + 2\sin t \cos t + \cos^2 t + \cos^2 t - 2\sin t \cos t + \sin^2 t}$$

$$= e^t\sqrt{1+1+1} = \sqrt{3}e^t$$

$$|\vec{a}| = \sqrt{e^{2t} + 4e^{2t}\,cos^2 t + 4e^{2t}\,sin^2 t}$$

$$|\vec{a}| = e^t\sqrt{1 + 4(cos^2 t + sin^2 t)} = \sqrt{5}e^t$$

$$cos\,\theta = \frac{e^{2t} + (e^t sin\,t + e^t\,cos\,t)e^{2t}\,cos\,t - (e^t\,cos\,t - e^t\,sin\,t)e^{2t}\,sin\,t}{\sqrt{3}e^t * \sqrt{5}e^t}$$

$$= \frac{e^{2t}(1 + 2\,sin\,t\,cos\,t + 2\;cos^2 t - 2\,sin\,t\,cos\,t + 2\,sin^2 t)}{\sqrt{15}e^{2t}}$$

$$= \frac{1 + 2(\,cos^2 t + sin^2 t)}{\sqrt{15}} = \frac{1+2}{\sqrt{15}} = \frac{3}{\sqrt{15}}$$

$$\theta = cos^{-1}\left(\frac{3}{\sqrt{15}}\right) = 39.2^0$$

Example(4):

Find the speed and length of the curve

$$\vec{R} = e^t\,cos\,t\,i + e^t sin\,t\,j + e^t k \quad from \quad t = 0 \;\; to \quad t = \pi$$

Sol.

$$\vec{V} = \frac{d\vec{R}}{dt} = (-\,e^t sin\,t + e^t\,cos\,t)i + (e^t\,cos\,t + e^t sin\,t)j + e^t k$$

$$speed = \vec{V} = \sqrt{(-\,e^t sin\,t + e^t\,cos\,t)^2 + (e^t\,cos\,t + e^t sin\,t)^2 + e^{2t}}$$

$$= \sqrt{e^{2t} sin^2 t - 2e^{2t}\,sin\,t\,cos\,t + e^{2t}\,cos^2 t + e^{2t}\,cos^2 t + 2e^{2t}\,sin\,t\,cos\,t + e^{2t}\,sin^2 t + e^{2t}}$$

$$=e^t\sqrt{1+1+1} = \sqrt{3}e^t$$

$$Length\ of\ curve = S = \int_0^{\pi} |\vec{V}| dt = \int_0^{\pi} \sqrt{3}e^t\, dt = \sqrt{3}\int_0^{\pi} e^t\, dt$$

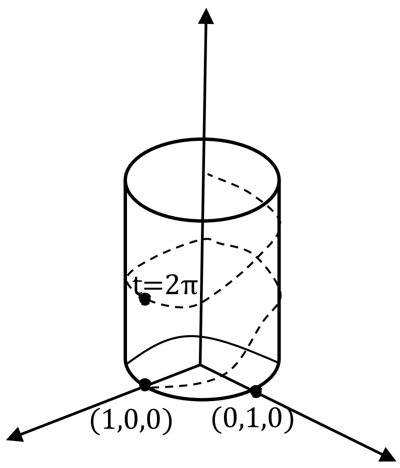
$$= \sqrt{3}[e^t]_0^{\pi} \quad = \sqrt{3}(e^{\pi} - 1)$$

Example(5):

Find the length of arc of one turn of the helix $\vec{R} = \cos t\, i + \sin t\, j + tk$

Note:

Since $\vec{R}(t) = xi + yj + zk$, the projection of the point P(x, y, z) onto xy-plane moves around the circle $x^2 + y^2 = 1$, z=0 as varies while the distance between ρ and the xy-plan changes steadily with t since cost and sint have period of 2π, the helix makes one full turn as t varies from 0 to 2π



Sol.

$$\vec{V} = \frac{\overrightarrow{dR}}{dt} = -\sin t\, i + \cos t\, j + k$$

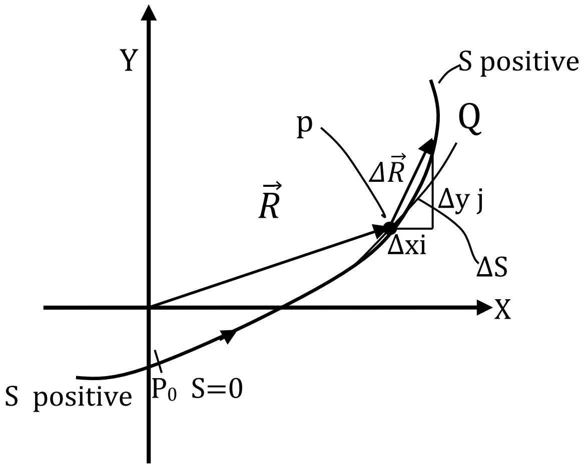
$$S = \int_0^{2\pi} |\vec{V}| dt = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + 1}\, dt$$

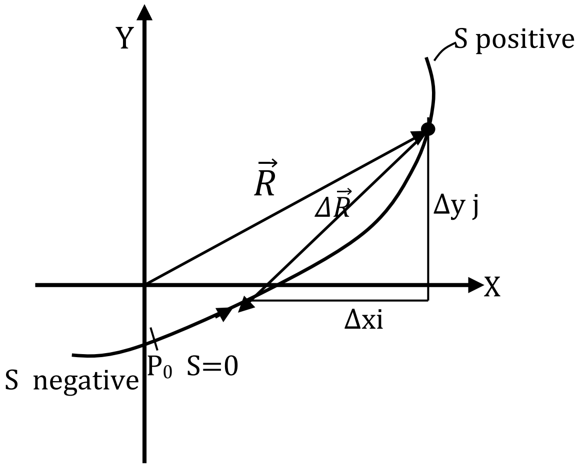
$$\int_0^{2\pi} \sqrt{2}\, dt = \sqrt{2}[t]_0^{2\pi} \quad = 2\sqrt{2}\pi$$

Tangetial Vector

Let point moves along a given curve in the xy-plane and its specified by the length of arc S from an arbitrary chosen reference point P_0 on the curve.

a)ΔS>0 b)ΔS<0





The vector $\vec{R} = xi + yj$ from the origin to point P is a function of S. Let P have coordinates (x, y) corresponding to the curve of S, while Q(x+Δx, y+Δy) corresponding to S.

$$\overrightarrow{\Delta R} = \Delta xi + \Delta yj$$

$$\frac{\overrightarrow{\Delta R}}{\Delta S} = \frac{\Delta x}{\Delta S} i + \frac{\Delta y}{\Delta S} j = \frac{\overrightarrow{PQ}}{\Delta S}$$

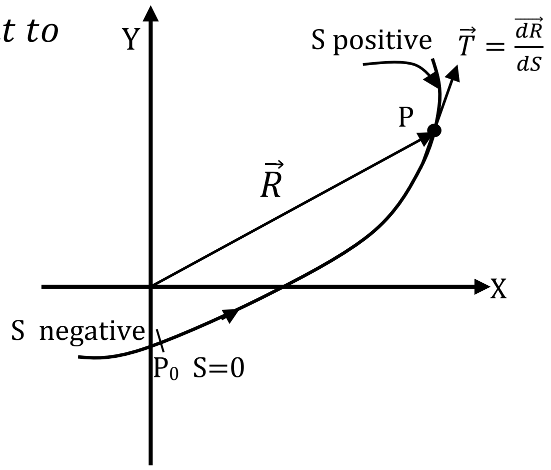
$\frac{\overrightarrow{\Delta R}}{\Delta S}$ is a vector whose magnitude is the chard PQ divided by the arc PQ and this approaches unity as ΔS→0. Hence $\frac{d\vec{R}}{dS} = \lim_{\Delta S \to 0} \frac{\Delta \vec{R}}{\Delta S}$ is a unit vector.

The vector $\frac{\Delta \vec{R}}{\Delta S}$ will point in the direction of increasing S in both figures a and b.

As $\Delta S \to 0$ and $Q \to P$, the direction of the tangent to the curve at P.

Thus the limiting divation $\frac{\Delta \vec{R}}{\Delta S}$ (the direction of $\vec{T} = \frac{d\vec{R}}{dS}$ is along the tangent to the curve at P).

When S is postive, $\vec{T}$ points away from P_o when S is negative, , $\vec{T}$ points toward P_0 , $\vec{T} = \frac{d\vec{R}}{dS}$ *is an unit vector tangent to the curve at point P* $\quad \vec{T} = \frac{d\vec{R}}{dS} = \frac{dx}{dS}i + \frac{dy}{dS}j$



The natural pararetrization of motion in many cases is likely to be time not arc length .

if $\vec{R} = x(t)i + y(t)j$,

then the best way to find $\vec{T}$ is to normalize $\vec{V} = \frac{\overrightarrow{dR}}{dt}$,

then if we find $\vec{V}$and then devide $\vec{V}$ by$|\vec{V}|$ to obtain $\vec{T} = \frac{\vec{V}}{|V|}$

<u>Note</u> $\vec{V} = \frac{d\vec{R}}{dt}$ *and* $|\vec{V}| = \frac{d\vec{S}}{dt}$

$$\vec{T} = \frac{\vec{V}}{|V|} = \frac{\frac{d\vec{R}}{dt}}{\frac{d\vec{S}}{dt}} = \frac{d\vec{R}}{dS}$$

<u>Example:</u>

If $\vec{R} = xi + yj = (e^t \cos t)i + (e^t \sin t)j$ is the vector from the origin to P(x, y). for this motion find the unit tangent vector $\vec{T}$

<u>Sol.</u>

$$\vec{V} = \frac{d\vec{R}}{dt} = (e^t \cos t - e^t \sin t)i + (e^t \sin t + e^t \cos t)j$$

$\vec{V} = e^t(\cos t - \sin t)i + e^t(\sin t + \cos t)j$

$|\vec{V}| = \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2}$

$=$

$e^t\sqrt{\cos^2 t + \sin^2 t - 2\sin t\cos t + \sin^2 t + \cos^2 t + 2\sin t\cos t}$

$= \sqrt{2}e^t$

$$\vec{T} = \frac{\vec{V}}{|V|} = \frac{e^t[(\cos t - \sin t)i + (\sin t + \cos t)j]}{\sqrt{2}e^t}$$

$$= \frac{(\cos t - \sin t)i + (\sin t + \cos t)j}{\sqrt{2}}$$

Example:

If $\vec{R} = xi + yj + zk$, find the unit tangent vector $\vec{T}$ for the space curve $x = 3\cosh 2t$, $y = 3\sinh 2t$, $z = 6t$.

Sol.

$\vec{R} = 3\cosh 2t\, i + 3\sinh 2t\, j + 6tk$

$$\vec{V} = \frac{d\vec{R}}{dt} = 6\sinh 2t\, i + 6\cosh 2t\, j + 6k$$

$|\vec{V}| = \sqrt{(6\sinh 2t)^2 + (6\cosh 2t)^2 + 6^2}$

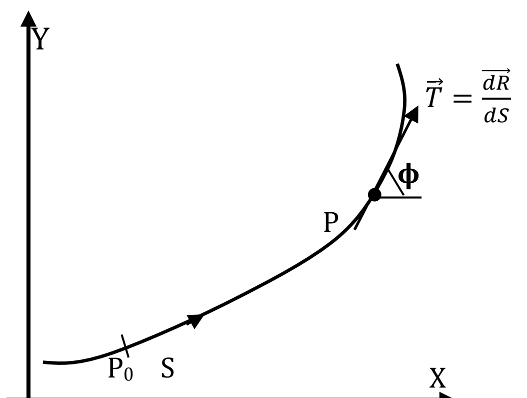
$= 6\sqrt{\sinh^2 3t + \cosh^2 2t + \cosh^2 2t - \sinh^2 2t} = 6\sqrt{2}\cosh 2t$

$$\vec{T} = \frac{\vec{V}}{|\vec{V}|} = \frac{6\sinh 2t\, i + 6\cosh 2t\, j + 6k}{6\sqrt{2}\cosh 2t}$$

$\vec{T} = \frac{1}{\sqrt{2}}(\tanh 2t\, i + j + \operatorname{sech} 2t\, k)$

Curvature and normal vectors

If P moves along a curve in the xy-plane,



we measure the direction of $\vec{T}$ by the angle **ϕ**

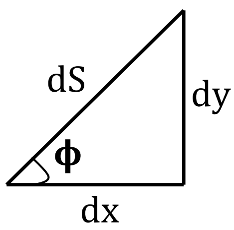
from the positively direction x-axis to $\vec{T}$.

(Note the length of $\vec{T}$ is constant, always

being equal to one. But the direction of $\vec{T}$ changes).

The rate at which $\vec{T}$ turns to one side or the other may be measured as we move along the curve by keeping track of the change in **ϕ**. The absolute value of the derivative $\frac{d\phi}{dS}$ of **ϕ** with respect to the arc. Length S, is called the curvature function, k, of the curve.

$k = \left|\frac{d\phi}{dS}\right|$



Where $tan\,\phi = \frac{dy}{dx}$

And $dS = \pm\sqrt{dx^2 + dy^2}$

Or $\phi = tan^{-1}\frac{dy}{dx}$

$$\frac{d\phi}{dx} = \frac{\frac{d^2y}{dx^2}}{1 + (\frac{dy}{dx})^2}$$

$$\frac{dS}{dx} = \pm\frac{\sqrt{dx^2 + dy^2}}{dx} = \pm\sqrt{1 + (\frac{dy}{dx})^2}$$

$$k = \left|\frac{d\phi}{dS}\right| = \left|\frac{d\phi/dx}{dS/dx}\right| = \frac{\frac{d^2y}{dx^2}/1+(\frac{dy}{dx})^2}{\pm\sqrt{1+(\frac{dy}{dx})^2}}$$

$$k = \frac{\frac{d^2y}{dx^2}}{\left[1 + (\frac{dy}{dx})^2\right]^{3/2}}$$

If x=f(t) and y=g(t)

$$k = \left|\frac{d\phi}{dS}\right| = \left|\frac{d\phi/dt}{dS/dt}\right|$$

$$\phi = tan^{-1}\left(\frac{dy}{dx}\right) = tan^{-1}\left(\frac{dy/dt}{dx/dt}\right)$$

$$\frac{d\phi}{dt} = \frac{1}{1+\left(\frac{dy/dt}{dx/dt}\right)^2} * \frac{\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^2}$$

$$dS = \mp\sqrt{dx^2 + dy^2} \; or \; \frac{dS}{dt} = \mp\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}$$

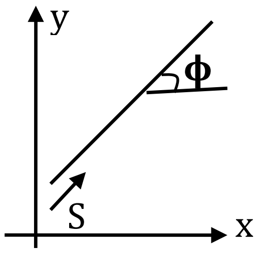
$$k = \left|\frac{\frac{1}{1+\left(\frac{dy/dt}{dx/dt}\right)^2} * \frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\mp\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}}\right|$$

$$k = \left|\frac{(\frac{dx}{dt})^2}{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} * \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\mp\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2(\frac{dx}{dt})^2}}\right|$$

$$k = \frac{\left|\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}\right|}{\left[(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2\right]^{3/2}}$$

Note :the curvature of a straight line is zero.

Since **ϕ** is constant, therefore $\frac{d\phi}{dS} = 0$.



Find the curvature of the curves

Example(1):

$$y = ln\,(cos\,x) \quad \rightarrow \quad \frac{dy}{dx} = \frac{1}{cos\,x}(-\,sin\,x) = -\,tan\,x$$

$$\frac{d^2y}{dx^2} = -\,sec^2\,x$$

$$k = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}} = \frac{|-\,sec^2\,x|}{[1+(-\,tan\,x)^2]^{3/2}} = \frac{|-\,sec^2\,x|}{[1+tan^2\,x]^{3/2}}$$

$$k = \frac{|-\,sec^2\,x|}{[sec^2\,x]^{3/2}} = \frac{|-\,sec^2\,x|}{sec^3\,x} = \left|\frac{-1}{sec\,x}\right| = |-cos\,x|$$

$$k = |cos\,x|$$

Example(2):

$$x = a(cos\,\theta + \theta\,sin\,\theta) \qquad y = a(sin\,\theta - \theta\,cos\,\theta)$$

$$\frac{dx}{d\theta} = a(-\,sin\,\theta + sin\,\theta + \theta\,cos\,\theta) \quad \rightarrow \quad \frac{dx}{d\theta} = a\,\theta\,cos\,\theta$$

$$\frac{d^2x}{d\theta^2} = a(cos\,\theta - \theta\,sin\,\theta)$$

$$\frac{dy}{d\theta} = a(cos\,\theta - cos\,\theta + \theta\,sin\,\theta) \quad \rightarrow \quad \frac{dy}{d\theta} = a\,\theta\,sin\,\theta$$

$$\frac{d^2y}{d\theta^2} = a(sin\,\theta + \theta\,cos\,\theta)$$

$$k = \frac{\left|\frac{dx}{d\theta}\,\frac{d^2y}{d\theta^2} - \frac{dy}{d\theta}\,\frac{d^2x}{d\theta^2}\right|}{\left[(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2\right]^{3/2}}$$

$$k = \frac{a\,\theta\cos\theta * a(\sin\theta + \theta\cos\theta) - a\,\theta\sin\theta * a(\cos\theta - \theta\sin\theta)}{[(a\,\theta\cos\theta)^2 + (a\,\theta\sin\theta)^2]^{3/2}}$$

$$k = \frac{|a^2\,\theta^2(\sin^2\theta + \cos^2\theta)|}{a^3\,\theta^3} = \frac{1}{a\theta}$$

<u>Example(3):</u>

$X = f(y) = \ln(\sec y)$

<u>Sol.</u>

$\frac{dx}{dy} = \frac{1}{\sec y}\ \sec y\ \tan y = \tan y$

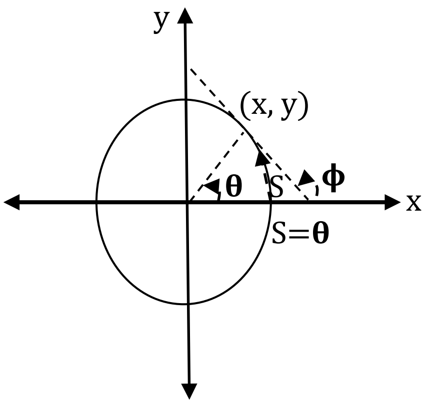
$$\frac{d^2x}{dy^2} = \sec^2 y$$

$$k = \frac{\frac{d^2x}{dy^2}}{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}} = \frac{|\sec^2 y|}{[1+\tan^2 y]^{3/2}}$$

$$k = \frac{|\sec^2 y|}{\sec^3 y} = \left|\frac{1}{\sec y}\right| = |\cos y|$$

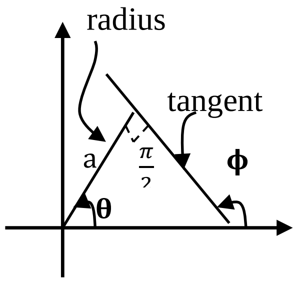
<u>Example(4):</u>

Find the curvature of a circle of radius a



$S = a\theta \quad \boldsymbol{\phi = \theta + \frac{\pi}{2}}$

$\frac{d\phi}{dS} = \frac{1}{a} \quad \rightarrow \quad k = \frac{d\phi}{dS} = \frac{1}{a}$



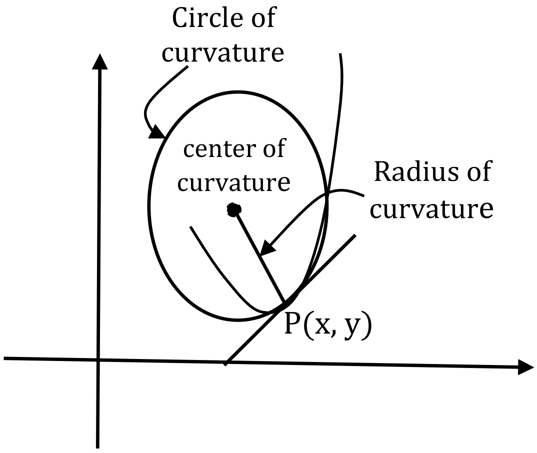
The curvature of circle is equal to the reciprocal of its radius,

the smaller is the circle, the greater is its curvature.

Radius of curvature

The circle that is tangent to a curve at point P(x, y) whose center lies on the concave side of the curve and that has the same curvature as the curve has at P, is called the circle of curvature, from the preceding example, its radius is $\frac{1}{k}$. This radius is defined as radius of curvature, ρ, at point P.

$$\rho = \frac{1}{k} = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{dy^2}{d^2x}\right|}$$



The circle of curvature lies toward

the inner side of the curve.

Unit normal vector

In terms of the slope angle **ϕ**, we may write

$$\vec{T} = i\cos\phi + j\sin\phi \qquad \ldots..(1)$$

$$\frac{d\vec{T}}{d\phi} = -i\sin\phi + j\cos\phi \qquad \ldots..(2)$$

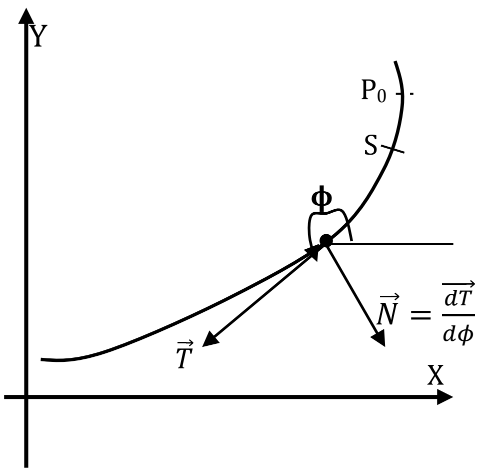
The above derivatives has the magnitude

$$\left|\frac{d\vec{T}}{d\phi}\right| = \sqrt{\sin^2\theta + \cos^2\theta} = 1$$

$$\vec{T}.\frac{\overrightarrow{dT}}{d\phi} = -\sin\phi\cos\phi + \cos\phi\sin\phi = 0$$

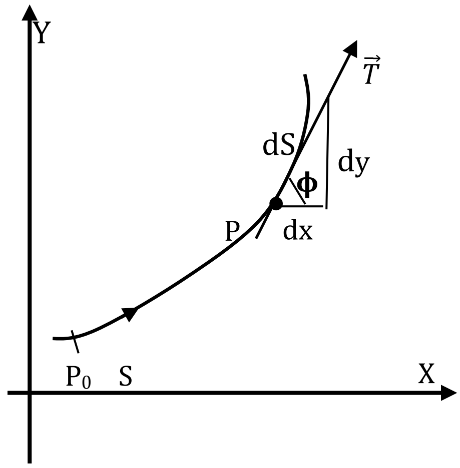
Therefore, $\frac{d\vec{T}}{d\phi}$ is normal to $\vec{T}$

$$\vec{N} = \frac{d\vec{T}}{d\phi} = -i\cos(\phi + 90) + j\sin(\phi + 90)$$



$\vec{N} = -i \sin\phi + j \cos\phi$

The vector $\vec{N}$ is called the unit normal vector obtained by rotating the unit tangent vector $\vec{T}$ counter clock wise through 90^0. A comparison of Eq.(1) and (2) shows that $\vec{N}$ can be found from $\vec{T}$ by inter changing components and changing the sign of the new first component.



Note $\vec{T} = \frac{dx}{dS} i + \frac{dy}{dS} j$ or

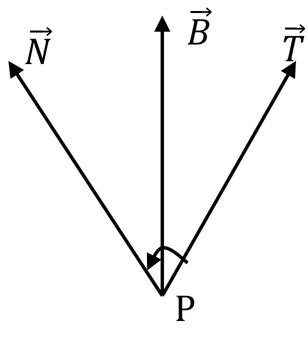
$\vec{T} = \cos\phi \; i + \sin\phi \; j$

Where $\cos\phi = \frac{dx}{dS}$ and $\sin\phi = \frac{dy}{dS}$

$\vec{N} = -\frac{dy}{dS} i + \frac{dx}{dS} j$

Once the vectors $\vec{T}$ and $\vec{N}$ have been determined, we may define a thired unit vector perpendicular to both $\vec{T}$ and $\vec{N}$ as

$\vec{B} = \vec{T} \times \vec{N}$



This vector is known as the binormal vector at point P.

$\vec{T}$, $\vec{N}$ and $\vec{B}$ form a right handed coordinate system.

Example:

If $\vec{R} = xi + yj$ and $x = 2t + 5$, $y = t^2 - 5$, find the unit tangent vector $\vec{T}$ and the unit normal vector $\vec{N}$.

Sol.

$\vec{R} = (2t + 5)i + (t^2 - 5)j$

$$\vec{V} = \frac{d\vec{R}}{dt} = 2i + 2tj$$

$$\left|\vec{V}\right| = \sqrt{4 + 4t^2} = 2\sqrt{1 + t^2}$$

$$\vec{T} = \frac{\vec{V}}{|V|} = \frac{2}{2\sqrt{1 + t^2}}i + \frac{2t}{2\sqrt{1 + t^2}}j$$

$$\vec{T} = \frac{1}{\sqrt{1+t^2}}i + \frac{t}{\sqrt{1+t^2}}j$$

$$\vec{N} = -\frac{t}{\sqrt{1 + t^2}}i + \frac{1}{\sqrt{1 + t^2}}j$$

H.W. For the data given in the preceding example find the curvature k